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Probabilistic Uncertain Linguistic T-Spherical Fuzzy Aggregation Operators-based Multi-Criteria Decision-Making Method and Its Application with Unknown Weight Information

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ABSTRACT

To comprehensively and precisely capture the fuzziness, uncertainty, hesitancy, and inherent randomness of evaluation information, this paper introduces a novel generalized concept: the probabilistic uncertain linguistic T-spherical fuzzy set (PULTSFS). This concept is developed by leveraging the unique advantages of both probabilistic uncertain linguistic term set (PULTS) and T-spherical fuzzy set (TSFS). The study begins by proposing a standardization method for PULTSFS, defining its score and accuracy functions, and establishing a Hamming distance measure for any two probabilistic uncertain linguistic T-spherical fuzzy numbers (PULTSFNs). Building upon the fundamental operational rules of PULTSFNs, the research then develops and analyzes the properties of weighted average and weighted geometric aggregation operators (AOs) tailored for PULTSFS. A decision-making framework using these AOs is subsequently constructed to address multi-criteria decision-making (MCDM) challenges where criteria weights are entirely unknown. Finally, the practicality and efficacy of the proposed method are validated through a numerical example, which includes a sensitivity analysis and comparative benchmarks against existing methods. The results demonstrate that the proposed method significantly improves the accuracy and reliability of resolving complex decision-making scenarios, ensuring that the resulting decisions are more practical and better aligned with real-world requirements.

1. Introduction

The MCDM, as a crucial component of modern decision science, is being applied across numerous domains in real life. How to accurately characterize the fuzziness and hesitancy in evaluation information has become a key focus in academic research.To overcome the limitations of classical fuzzy sets that only contain membership information[1], Atanassov[2] proposed the concept of intuitionistic fuzzy set (IFS), which can simultaneously account for both membership and non-membership information. However, the IFS is constrained by the condition that the sum of membership and non-membership degrees must not exceed 1. To overcome this limitation,Yager[3] introduced Pythagorean fuzzy set (PyFS), which allows the sum of the squares of membership and

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non-membership degrees to be less than or equal to 1. This enables PyFS to possess stronger capabilities in representing fuzzy information compared to IFS. What is more, Yager[4] proposed a more generalized concept of q-rung orthopair fuzzy set (q-ROFS), which serves as an extension of both IFS and PyFS. By relaxing the constraint to allow the sum of the q-th powers of membership and non-membership degrees to be less than or equal to 1, the q-ROFS gains the capability to represent a broader range of fuzzy information while maintaining strong operational properties. However, in certain decision-making scenarios, individuals expressing opinions may not only provide "affirmative" or "negative" information but also convey some degree of "abstain" information. To address such situations, Cuong and Kreinovich[5] introduced picture fuzzy set (PFS), which requires that the sum of membership, abstain, and non-membership degrees does not exceed 1. Mahmood et al.[6] further extended this framework by proposing the concepts of spherical fuzzy set (SFS) and TSFS, where the TSFS satisfies the condition that the sum of the q-th powers of membership, neutrality, and non-membership degrees is less than or equal to 1 ($q \geq 1$). Compared with PFS and SFS, the TSFS enables experts to express their opinions on evaluation objects in a more comprehensive and flexible manner, providing greater freedom for expert judgment. Consequently, it is recognized as a more powerful tool for effectively characterizing and processing fuzzy information[7,8,9] .

Existing derived fuzzy set theories can only handle decision-making problems with uncertainty from a quantitative perspective. However, when confronted with complex uncertain decision-making scenarios, individuals often find it difficult to express their viewpoints using precise values based solely on personal experience and cognitive processes. Qualitative linguistic evaluations such as "favorable prospects," "high expected returns," or "significant risks" better align with human intuition. Zadeh[10] proposed the concept of Linguistic Term Sets (LTS), which utilizes flexible words or phrases consistent with human descriptive habits to represent decision information. However, in practical decision-making processes, a single linguistic variable may be insufficient to fully capture the descriptive information provided by DMs. To address this, Rodríguez et al.[11] introduced the concept of Hesitant Fuzzy Linguistic Term Sets (HFLTS), allowing DMs to use multiple linguistic terms simultaneously to express their evaluations. Nevertheless, the linguistic terms contained in HFLTS are assigned equal importance, failing to reflect DMs' preferences for specific terms. To better represent the fuzziness, uncertainty, and hesitancy in decision information, Pang et al.[12] proposed the concept of Probabilistic Linguistic Term Sets (PLTS) and established its fundamental operational rules.A PLTS consists of all possible linguistic terms and their corresponding probabilities. As an extension of HFLTS, it effectively prevents the loss of original evaluation information and provides a more accurate description of group evaluation data. In practical decision-making processes, due to differences in expertise and backgrounds, people often exhibit cognitive ambiguity when evaluating objects, making them more inclined to use uncertain linguistic terms for decision analysis. To address this, Lin et al.[13] extended PLTS and proposed PULTS. This framework effectively captures the fuzziness, hesitancy, and uncertainty in decision-making processes, offering a more refined modeling tool for real-world ambiguous decision problems[14] Currently, the PULTS is widely applied to solve problems such as technology route selection[14], supplier selection[15], teaching quality evaluation[16] and production risk assessment[17].Since its introduction, scholars have further extended and generalized this concept for different scenarios. For example, Gong and Chen[18] proposed the concept of Probabilistic Uncertain Linguistic Intuitionistic Fuzzy Sets (PULIFS), and developed a series of goal programming models, applying this methodology to a virtual reality project selection case study. Naz et al.[19] proposed a novel concept integrating the advantages of PULTS and q-ROFS, PULq-ROFS, and developed a CRITIC-VIKOR method to address industrial robot selection problems. Although PULq-ROFS offers broader information representation capabilities compared to PULIFS and PULTS, it characterizes evaluation information solely through the two dimensions of membership and non-membership degrees, failing to capture abstain degrees in evaluation information. Therefore, in addressing complex evaluation and decision-making problems, there is an urgent need to develop a new tool capable of expressing more comprehensive and extensive evaluation information.

Based on the aforementioned literature review, no integrated research combining PULTS with TSFS has been reported to date, nor have corresponding MADM methods been proposed. Therefore, this paper introduces a novel concept of PULTSFS and establishes its associated operational rules, building upon the respective advantages of PULTS and TSFS. The primary objective of this extension is to mitigate potential limitations in characterizing the fuzzy and uncertain information provided by DMs when using PULTS or TSFS independently. Based on the above analysis, the research motivations of this study can be summarized as follows:

(1) In practical scenarios, evaluating and selecting the optimal alternative constitutes a complex MCDM problem characterized by multiple uncertainties. Existing decision-making techniques that rely solely on PULTS or TSFS to represent DMs' evaluation information may lead to partial information distortion. Therefore, it is imperative to establish a more generalized and universally applicable model to deeply and comprehensively characterize and explore the potential value of alternatives.

(2) Compared with existing decision-making models, PULTSFS demonstrates greater flexibility and practicality. This characteristic enables it to yield more precise results when addressing MADM problems.

(3) The AOs proposed in this paper effectively overcome the shortcomings and limitations of existing AOs. Their universality makes them applicable not only to probabilistic uncertain linguistic T-spherical fuzzy (PULTSF) information but also to various existing fuzzy derivative sets such as PULq-ROFS, PULIFS, PLTSFS, and PLq-ROFS.

The main contributions of this paper include:

(1) The novel concept of PULTSFS is proposed, and its basic operational rules and Hamming distance measure are defined.

(2) Two novel types of AOs are innovatively developed for fusing PULTSF information, namely the PULTSF weighted averaging (PULTFWA), PULTSF ordered weighted averaging (PULTSFOWA), PULTSF weighted geometric (PULTSFWG) and PULTSF ordered weighted geometric (PULTSFOWG) operators, along with discussing their relevant properties.

(3) Establishing a comprehensive PULTSF MCDM technical framework based on the proposed AOs.

(4) Validating the effectiveness of the proposed methodology through numerical example, while demonstrating its practical value and advantages via sensitivity and comparative analyses.

The structure of this paper is organized as follows: Section 2 provides a brief introduction to the fundamental concepts of LTS, ULVs, PLTS, PULTS, and TSFS; Section 3 defines the novel concept of PULTSFS; Section 4 develops a series of PULTSF AOs; an AOs-based MCDM method with PULTSF information is designed in Section 5; Section 6 presents a numerical example; and Section 7 concludes the study and outlines future research directions.:

2. Preliminaries

Some basic concepts, including LTS, ULVs, PLTS, PULTS, and TSFS, are involved as the theoretical basis for the novel PULTSFS concept introduced in this paper.

2.1 Figure Style and Format

In order to better adapt to the subjective feelings of decision makers on the evaluation object, Herrera et al.[20] defined a set of discrete itemsets containing an odd number of finite and completely ordered as linguistic itemsets, that is $S_{[0,k-1]} = \{s_0, s_1, \cdots s_{k-1}\}$[20].

In general, k can take values of 7 or 9. For example, when k = 7, the language item set has 7 linguistic items, which can be represented as $S_{[0,6]} = \{s_0, s_1, \cdots s_6\} = \{\text{very low}, \text{low}, \text{relatively low}, \text{medium}, \text{relativey high}, \text{high}, \text{very high}\}$.

In general, for any linguistic itemset $S_{[0,k-1]}$, s_α and s_β should satisfy the following conditions[20] :

(1) The set is ordered : $s_\alpha < s_\beta$, if and only if $\alpha < \beta$;

(2) There is a negative operator : neg(s_α)=s_β, such that $\beta=2k-\alpha$;

(3) If $\alpha \leq \beta$, then max{s_α, s_β}=s_β and min{s_α, s_β}=s_α.

Definition 1 [21]. On the basis of the set $\bar{S} = \{s_x | x \in R^+\}$ of continuous linguistic terms, assuming that $\tilde{s} = [s_\alpha, s_\beta], s_\alpha, s_\beta \in \bar{S}$ and $0 < \alpha \leq \beta$, s_α, s_β are the lower and upper bounds of $\tilde{s}$, respectively, then $\tilde{s}$ is called a uncertain linguistic value (ULV).

Due to the shortcomings of traditional ULVs operation rules, for example, the result of addition operation may exceed the upper limit of the predefined LTS S. Liu and Zhang[22] gave new ULVs operation rules as follows :

Definition 2 [22]. Assume that, $\tilde{s} = [s_\alpha, s_\beta]$, $\tilde{s}_1 = [s_{\alpha1}, s_{\beta1}]$, $\tilde{s}_2 = [s_{\alpha2}, s_{\beta2}]$ for any three ULVs, the constant $\lambda \geq 0$, then their operation rules are expressed as follows :

(1) $\tilde{s}_1 \oplus \tilde{s}_2 = \left[s_{\alpha1+\alpha2-\frac{\alpha1\times\alpha2}{k}}, s_{\beta1+\beta2-\frac{\beta1\times\beta2}{k}}\right]$,

(2) $\tilde{s}_1 \otimes \tilde{s}_2 = \left[s_{\frac{\alpha1\times\alpha2}{k}}, s_{\frac{\beta1\times\beta2}{k}}\right]$,

(3) $\lambda\tilde{s} = \left[s_{k-k\left(1-\frac{\alpha}{k}\right)^\lambda}, s_{k-k\left(1-\frac{\beta}{k}\right)^\lambda}\right]$,

(4) $(\tilde{s})^\lambda = \left[s_{k\left(\frac{\alpha}{k}\right)^\lambda}, s_{k\left(\frac{\beta}{k}\right)^\lambda}\right]$.

2.2 PLTS and PULTS

Pang et al.[12] defined the concept of PLTS for the first time, which is obtained by combining HFLTS with probabilistic information to preserve the given linguistic information to the greatest extent.

Definition 3 [12]. Let $S_{[0,k-1]}$ be an LTS, and a PLTS is defined as follows :

$$L(p)=\left\{s^{(t)}\left(p^{(t)}\right)\middle| s^{(t)}\in S_{[0,k\text{-}1]}, p^{(t)}\geq 0, t=1,2,\ldots,\#T, \sum_{t=1}^{\#T} p^{(t)}\leq 1\right\} \tag{1}$$

where $s^{(t)}\left(p^{(t)}\right)$ denotes the linguistic term associated with probability $p^{(t)}$, #T is the number of all different linguistic terms in $L(p)$.

In order to more accurately express the concerns of decision makers, Lin et al.[13] proposed the concept of PULTS based on ULV and PLTS.

Definition 4 [13]. A PULTS is defined as follows :

$$UL(p)=\left\{\left\langle\left[s_\alpha^{(t)}, s_\beta^{(t)}\right], p^{(t)}\right\rangle\middle| p^{(t)}\geq 0, t=1,2,\ldots,\#T, \sum_{t=1}^{\#T} p^{(t)}\leq 1\right\} \tag{2}$$

Among them, $\langle[s_\alpha^{(t)},s_\beta^{(t)}],p^{(t)}\rangle$ is the t-th PUL element in $UL(p)$, $s_\alpha^{(t)}$, $s_\beta^{(t)}$ are the lower and upper language terms, respectively, and $p^{(t)}$ is the corresponding probability.

Definition 5 [13]. Let $UL_1(p)=\{\langle[s_{\alpha1}^{(t)}, s_{\beta1}^{(t)}], p_1^{(t)}\rangle|p_1^{(t)}\geq0,t=1,2,\ldots,\#T_1\}$ and $UL_2(p)=\{\langle[s_{\alpha2}^{(t)}, s_{\beta2}^{(t)}], p_2^{(t)}\rangle|p_2^{(t)}\geq0\ ,t=1,2,\ldots,\#T_2\}$ be two PULTSs. For ease of calculation, they need to be normalized as follows :

(1) If $0<\sum_{t=1}^{\#T_j}p_j^{(t)}<1$, then $UL_j(p)$ is normalized to $UL_j(\tilde{p})=\left\{\langle[s_{\alpha j}^{(t)}, s_{\beta j}^{(t)}], \tilde{p}_j^{(t)}\rangle\left|\tilde{p}_j^{(t)}\geq0,t=1,2,\ldots,\#T_j,\sum_{t=1}^{\#T_j}\tilde{p}_j^{(t)}=1\right.\right\}$, where $\tilde{p}_j^{(t)}=p_j^{(t)}/\sum_{t=1}^{T_j}p_j^{(t)}$, j=1, 2.

(2) If $\#T_1\neq\#T_2$, then we need to add some linguistic terms to the one with fewer elements, the probability is 0, and the normalized PULTS is obtained.

2.3 The concept of TSFS

Compared with traditional fuzzy sets and their variants, TSFS can provide experts with a broader expression space and greater flexibility in three-dimensional aspects. Mahmood et al.[6] gave the following definition :

Definition 6 [6]. Assume that $X=\{x_1, x_2, \ldots, x_n\}$ is a domain, then a TSFS A on X is an object in the following form :

$$A=\{\langle x_j,(\mu_A(x_j),\eta_A(x_j),v_A(x_j))\rangle/x_j\in X\} \tag{3}$$

Among them, $\mu_A(x_j),\eta_A(x_j),v_A(x_j):X\rightarrow[0,1]$ represent the membership, neutrality and non-membership of x_j to A, respectively, and they satisfy the $0\leq(\mu_A(x_j))^q+(\eta_A(x_j))^q+(v_A(x_j))^q\leq1(q\geq1)$ condition. In order to facilitate the calculation, a triple $a_j=(\mu_j,\eta_j,v_j)$ is called T-spherical fuzzy number (TSFN), and the parameter q value can be set appropriately. When the q value is larger, the strength of the commitment is smaller ; the smaller the q value, the less hesitation, and the less uncertainty.

Definition 7 [23, 24]. Let $a_j=(\mu_j,\eta_j,v_j)(j=0,1,2)$ be three TSFNs, $\lambda>0$, then their algorithms are as follows :

(1) $a_1\oplus a_2=\left(\sqrt[q]{\mu_1^q+\mu_2^q-\mu_1^q\mu_2^q},\eta_1\eta_2,v_1v_2\right)$;

(2) $a_1\otimes a_2=\left(\mu_1\mu_2,\sqrt[q]{\eta_1^q+\eta_2^q-\eta_1^q\eta_2^q},\sqrt[q]{v_1^q+v_2^q-v_1^qv_2^q}\right)$;

(3) $\lambda a_0=\left(\sqrt[q]{1-(1-\mu_0^q)^\lambda},\eta_0^\lambda,v_0^\lambda\right)$;

(4) $(a_0)^\lambda=\left(\mu_0^\lambda,\sqrt[q]{1-(1-\eta_0^q)^\lambda},\sqrt[q]{1-(1-v_0^q)^\lambda}\right)$;

(5) $(a_0)^c=(v_0,\eta_0,\mu_0)$.

3. The basic concept of PULTSFS

Based on the advantages of PULTS and TSFS, we define a new type of probabilistic fuzzy set, PULTSFS, which not only allows experts to use multiple language items from three dimensions to express evaluation information, but also includes the possibility of each language item. The definition of PULTSFS is as follows :

Definition 8. Let X be a domain and $S_{[0,k-1]}$ be an LTS. Then define a PULTSFS $\widetilde{A}(p)$ on X as:

$$\widetilde{A}(p)=\left\{\langle x_\varsigma,\ \varphi_\varsigma(\hat{p})(x_\varsigma),\ \phi_\varsigma(\tilde{p})(x_\varsigma),\ \psi_\varsigma(\bar{p})(x_\varsigma)\rangle \middle| x_\varsigma \in X\right\} \quad (4)$$

where $\varphi_\varsigma(\hat{p})(x_\varsigma)=\left\{\left[s_{\mu^L_{\varsigma(t)}},\ s_{\mu^U_{\varsigma(t)}}\right](\hat{p}_{(t)}) \middle| s_{\mu^L_{\varsigma(t)}},\ s_{\mu^U_{\varsigma(t)}} \in S_{[0,\ k\text{-}1]},\ \hat{p}_{(t)}\geq 0,\ \sum_{t=1}^{\#T}\hat{p}_{(t)}\leq 1\right\}$ denotes the membership degree of $x_\varsigma\in X$; $\phi_\varsigma(\tilde{p})(x_\varsigma)=\left\{\left[s_{\eta^L_{\varsigma(r)}},\ s_{\eta^U_{\varsigma(r)}}\right](\tilde{p}_{(r)}) \middle| s_{\eta^L_{\varsigma(r)}},\ s_{\eta^U_{\varsigma(r)}} \in S_{[0,\ k\text{-}1]},\ \tilde{p}_{(r)}\geq 0,\ \sum_{r=1}^{\#R}\tilde{p}_{(t)}\leq 1\right\}$ denotes the neutrality degree of $x_\varsigma\in X$; $\psi_\varsigma(\bar{p})(x_\varsigma)=\left\{\left[s_{\nu^L_{\varsigma(w)}},\ s_{\nu^U_{\varsigma(w)}}\right](\bar{p}_{(w)}) \middle| s_{\nu^L_{\varsigma(w)}},\ s_{\nu^U_{\varsigma(w)}} \in S_{[0,\ k\text{-}1]},\ \bar{p}_{(w)}\geq 0,\ \sum_{w=1}^{\#W}\bar{p}_{(w)}\leq 1\right\}$ denotes the non-membership degree of $x_\varsigma\in X$, and the related probabilities are $\hat{p}_{(t)}$, $\tilde{p}_{(r)}$ and $\bar{p}_{(w)}$, respectively. For $x_\varsigma\in X$, they satisfy the condition $0\leq\left(max_{t=1}^{\#T}\mu^U_{\varsigma(t)}\right)^q+\left(max_{r=1}^{\#R}\eta^U_{\varsigma(t)}\right)^q+\left(max_{w=1}^{\#W}\nu^U_{\varsigma(w)}\right)^q\leq k^q$ $(q\geq 1)$.

If $X = \{ x \}$, then PULTSFS degenerates into a PULTSFN, that is,

$$\tilde{a}(p)=\langle\left\{\left[s_{\mu^L_{(t)}},\ s_{\mu^U_{(t)}}\right]|\hat{p}_{(t)}\right\},\left\{\left[s_{\eta^L_{(r)}},\ s_{\eta^U_{(r)}}\right]|\tilde{p}_{(r)}\right\},\left\{\left[s_{\nu^L_{(w)}},\ s_{\nu^U_{(w)}}\right]|\bar{p}_{(w)}\right\}\rangle \quad (5)$$

where $s_{\mu^L_{(t)}},\ s_{\mu^U_{(t)}},\ s_{\eta^L_{(r)}},\ s_{\eta^U_{(r)}},\ s_{\nu^L_{(w)}},\ s_{\nu^U_{(w)}}\in S_{[0,\ k\text{-}1]},\sum_{t=1}^{\#T}\hat{p}_{(t)}\leq 1,\sum_{r=1}^{\#R}\tilde{p}_{(t)}\leq 1,\sum_{w=1}^{\#W}\bar{p}_{(w)}\leq 1.$

Example 1. There is an LTS $S_{[0,\ 6]}$. Suppose two experts are invited to assess whether a company's potential value is worth investing in. The first expert believes that 30% of the investment is worth expressing opinions in the ULV $[s_3, s_4]$, and 15 % of the neutral attitude is expressed in the ULV $[s_2, s_4]$, and 40 % of the investment is not worth investing. Convinced and express opinions in the ULV $[s_1, s_2]$. Then the expert's evaluation information can be expressed as: $\tilde{a}_1(p)=\langle\{[s_3,s_4]|0.3\},\{[s_2,s_4]|0.15\},\{[s_1,s_2]|0.4\}\rangle$.

The second expert has some hesitation. He believes that 60 % of the determination is worth investing in the ULV $[s_2, s_3]$, and another 30 % is worth investing in the ULV $[s_2, s_4]$. At the same time, he holds a 35 % neutral attitude to express his opinion in the ULV $[s_3, s_5]$, and 50 % is not worth investing in the ULV $[s_3, s_4]$. Then the expert's evaluation information can be expressed as: $\tilde{a}_2(p)=\langle\{[s_2,s_3]|0.6,\ [s_2,s_4]|0.3,\ \},\{[s_3,s_5]|0.35\},\{[s_3,s_4]|0.5\}\rangle$.

Remark 1. For different values of parameter q, PULTSFS can be degraded into different specific forms, as follows :

When q = 1, $\widetilde{A}(p)$ is reduced to probabilistic uncertain linguistic PFS (PULPFS) ;

When q = 2, $\widetilde{A}(p)$ is reduced to a probabilistic uncertain linguistic SFS (PULSFS) ;

When $\phi_\varsigma(\tilde{p})(x_\varsigma)=0$, $\widetilde{A}(p)$ is reduced to a PULq-ROFS [19];

When q = 1, $\phi_\varsigma(\tilde{p})(x_\varsigma)=0$, $\widetilde{A}(p)$ is reduced to a PULIFS [18];

When q = 2, $\phi_\varsigma(\tilde{p})(x_\varsigma)=0$, $\widetilde{A}(p)$ is reduced to a probabilistic uncertain PyFS (PULPyFS) ;

When $s_{\mu^L_{(t)}}=s_{\mu^U_{(t)}}$,$s_{\eta^L_{(r)}}=s_{\eta^U_{(r)}}$ and $s_{\nu^L_{(w)}}=s_{\nu^U_{(w)}}$, $\widetilde{A}(p)$ is reduced to a PLTSFS [25];

When $\phi_\varsigma(\tilde{p})(x_\varsigma)=0$, $s_{\mu^L_{(t)}}=s_{\mu^U_{(t)}}$ and $s_{\nu^L_{(w)}}=s_{\nu^U_{(w)}}$, $\widetilde{A}(p)$ is reduced to a PLq-ROFS [26,27];

When q = 1, $\phi_\varsigma(\tilde{p})(x_\varsigma)=0$, $s_{\mu^L_{(t)}}=s_{\mu^U_{(t)}}$ and $s_{\nu^L_{(w)}}=s_{\nu^U_{(w)}}$, $\widetilde{A}(p)$ is reduced to a probabilistic linguistic IFS (PLIFS) ;

When q = 2, $\phi_\varsigma(\tilde{p})(x_\varsigma)=0$, $s_{\mu^L_{(t)}}=s_{\mu^U_{(t)}}$ and $s_{\nu^L_{(w)}}=s_{\nu^U_{(w)}}$, $\widetilde{A}(p)$ is reduced to a probabilistic linguistic PyFS (PLPyFS).

Obviously, PULTSFS has a strong generalization, and in some special cases, PULTSFS can be degraded to a special form.

Definition 9. Let any two PULTSFNs, $\tilde{a}_1(p) = \left\langle \left\{ \left[s_{\mu^L_{1(t)}}, s_{\mu^U_{1(t)}} \right] \middle| \hat{p}_{1(t)} \right\}, \left\{ \left[s_{\eta^L_{1(r)}}, s_{\eta^U_{1(r)}} \right] \middle| \tilde{p}_{1(r)} \right\}, \left\{ \left[s_{\nu^L_{1(w)}}, s_{\nu^U_{1(w)}} \right] \middle| \bar{p}_{1(w)} \right\} \right\rangle$, $\tilde{a}_2(p) = \left\langle \left\{ \left[s_{\mu^L_{2(t)}}, s_{\mu^U_{2(t)}} \right] \middle| \hat{p}_{2(t)} \right\}, \left\{ \left[s_{\eta^L_{2(r)}}, s_{\eta^U_{2(r)}} \right] \middle| \tilde{p}_{2(r)} \right\}, \left\{ \left[s_{\nu^L_{2(w)}}, s_{\nu^U_{2(w)}} \right] \middle| \bar{p}_{2(w)} \right\} \right\rangle$. For ease of calculation, they need to be normalized as follows :

(1) If $0 < \sum_{t=1}^{\#T} \hat{p}_{j(t)} < 1$ (take membership probability as an example, *j*=1,2), then $\tilde{a}_j(p)$ is normalized to $\tilde{a}_j(p^n)$, then the probability is $\hat{p}^n_{j(t)} = \hat{p}_{j(t)} / \sum_{t=1}^{\#T} \hat{p}_{j(t)}$, and the corresponding normalized PULTSFN can be described as:

$$\tilde{a}_j(p^n) = \left\langle \left\{ \left[s_{\mu^L_{j(t)}}, s_{\mu^U_{j(t)}} \right] \middle| \hat{p}^n_{j(t)} \right\}, \left\{ \left[s_{\eta^L_{j(r)}}, s_{\eta^U_{j(r)}} \right] \middle| \tilde{p}^n_{j(r)} \right\}, \left\{ \left[s_{\nu^L_{j(w)}}, s_{\nu^U_{j(w)}} \right] \middle| \bar{p}^n_{j(w)} \right\} \right\rangle \tag{6}$$

(2) If (for example, membership), then you need to add some linguistic terms to the less number of elements, the probability is 0, resulting in normalized PULTSFNs, (*j* = 1,2).

Example 2. Assume that there are two PULTSFNs, a and b, namely:

$$\tilde{a}_1(p) = \begin{pmatrix} \{[s_3, s_4]|0.2,[s_4, s_5]|0.6\}, \\ \{[s_1, s_2]|0.4,[s_2, s_4]|0.6\}, \\ \{[s_5, s_6]|0.3,[s_4, s_5]|0.7\} \end{pmatrix},$$

$$\tilde{a}_2(p) = \begin{pmatrix} \{[s_2, s_3]|0.4,[s_3, s_4]|0.2, [s_5, s_6]|0.2\}, \\ \{[s_4, s_5]|0.3,[s_2, s_3]|0.6\}, \\ \{[s_1, s_2]|0.3,[s_2, s_3]|0.4\} \end{pmatrix}.$$

Then, (1) according to Definition 9, we perform initial standardization on this :

$$\tilde{a}_1(p^n) = \begin{pmatrix} \{[s_3, s_4]|0.25,[s_4, s_5]|0.75\}, \\ \{[s_1, s_2]|0.40,[s_2, s_4]|0.60\}, \\ \{[s_4, s_5]|0.70,[s_5, s_6]|0.30\} \end{pmatrix},$$

$$\tilde{a}_2(p^n) = \begin{pmatrix} \{[s_2, s_3]|0.50,[s_3, s_4]|0.25, [s_5, s_6]|0.25\}, \\ \{[s_4, s_5]|0.33,[s_2, s_3]|0.67\}, \\ \{[s_1, s_2]|0.43,[s_2, s_3]|0.57\} \end{pmatrix}.$$

(2)

$$\tilde{a}_1(p^N) = \begin{pmatrix} \{[s_3, s_4]|0.00, [s_3, s_4]|0.25,[s_4, s_5]|0.75\}, \\ \{[s_1, s_2]|0.40,[s_2, s_3]|0.30, [s_3, s_4]|0.30\}, \\ \{[s_4, s_5]|0.7 ,[s_5, s_6]|0.3 \} \end{pmatrix},$$

$$\tilde{a}_2(p^N) = \begin{pmatrix} \{[s_2, s_3]|0.50,[s_3, s_4]|0.25, [s_5, s_6]|0.25\}, \\ \{[s_2, s_3]|0.00, [s_2, s_3]|0.67, [s_4, s_5]|0.33\}, \\ \{[s_1, s_2]|0.43,[s_2, s_3]|0.57\} \end{pmatrix}.$$

Definition 10. Assume that $S_{[0,k-1]}$ is an LTS, for any PULTSFN $\tilde{a}(p) = \left\langle \left\{ \left[s_{\mu^L_{(t)}}, s_{\mu^U_{(t)}} \right] \middle| \hat{p}_{(t)} \right\}, \left\{ \left[s_{\eta^L_{(r)}}, s_{\eta^U_{(r)}} \right] \middle| \tilde{p}_{(r)} \right\}, \left\{ \left[s_{\nu^L_{(w)}}, s_{\nu^U_{(w)}} \right] \middle| \bar{p}_{(w)} \right\} \right\rangle$, where $s_{\mu^L_{(t)}}, s_{\mu^U_{(t)}}, s_{\eta^L_{(r)}}, s_{\eta^U_{(r)}}, s_{\nu^L_{(w)}}, s_{\nu^U_{(w)}} \in S_{[0,k-1]}$, (t=1,2,...,#T; r=1,2,...,#R; w=1,2,...,#W), the score function of $\tilde{a}(p)$ is defined as:

$$Sc(\tilde{a}(p)) = s_{\frac{1}{2}\left(1 + \left(\frac{\sum_{t=1}^{\#T}\left(\frac{1}{2}\left(\mu^L_{(t)} + \mu^U_{(t)}\right)\hat{p}_{(t)}\right)}{k\sum_{t=1}^{\#T}\hat{p}_{(t)}}\right)^q - \left(\frac{\sum_{r=1}^{\#R}\left(\frac{1}{2}\left(\eta^L_{(t)} + \eta^U_{(t)}\right)\tilde{p}_{(r)}\right)}{k\sum_{r=1}^{\#R}\tilde{p}_{(r)}}\right)^q - \left(\frac{\sum_{w=1}^{\#W}\left(\frac{1}{2}\left(\nu^L_{(t)} + \nu^U_{(t)}\right)\bar{p}_{(w)}\right)}{k\sum_{w=1}^{\#W}\bar{p}_{(w)}}\right)^q\right)} \tag{7}$$

The accuracy function of $\tilde{a}(p)$ is defined as:

$$Ac(\tilde{a}(p)) = s_{\left(\frac{\sum_{t=1}^{\#T}\left(\frac{1}{2}\left(\mu^L_{(t)} + \mu^U_{(t)}\right)\hat{p}_{(t)}\right)}{k\sum_{t=1}^{\#T}\hat{p}_{(t)}}\right)^q + \left(\frac{\sum_{r=1}^{\#R}\left(\frac{1}{2}\left(\eta^L_{(t)} + \eta^U_{(t)}\right)\tilde{p}_{(r)}\right)}{k\sum_{r=1}^{\#R}\tilde{p}_{(r)}}\right)^q + \left(\frac{\sum_{w=1}^{\#W}\left(\frac{1}{2}\left(\nu^L_{(t)} + \nu^U_{(t)}\right)\bar{p}_{(w)}\right)}{k\sum_{w=1}^{\#W}\bar{p}_{(w)}}\right)^q} \tag{8}$$

Definition 11. Assume that there are two arbitrary sum of PULTSFNs, then their comparison rules are as follows :

(1) If $Sc(\tilde{a}_1(p)) \succ Sc(\tilde{a}_2(p))$, then $\tilde{a}_1(p) \succ \tilde{a}_2(p)$;

(2) If $Sc(\tilde{a}_1(p)) \prec Sc(\tilde{a}_2(p))$, then $\tilde{a}_1(p) \prec \tilde{a}_2(p)$;

(3) If $Sc(\tilde{a}_1(p)) = Sc(\tilde{a}_2(p))$, then

If $\mathrm{Ac}(\tilde{a}_1(p)) \succ \mathrm{Ac}(\tilde{a}_2(p))$, then $\tilde{a}_1(p) \succ \tilde{a}_2(p)$;

If $Ac(\tilde{a}_1(p)) \prec Ac(\tilde{a}_2(p))$,then $\tilde{a}_1(p) \prec \tilde{a}_2(p)$;

If $Ac(\tilde{a}_1(p)) = Ac(\tilde{a}_2(p))$, then $\tilde{a}_1(p) \approx \tilde{a}_2(p)$.

Example 3. Taking two PULTSFNs $\tilde{a}_1(p)$ and $\tilde{a}_2(p)$ in Example 2 as examples, their score function values are calculated and compared. According to Eq. (7), q = 4, then

$$Sc(\tilde{a}_1(p)) = s_{\frac{1}{2}\left(\begin{array}{c}1+\left(\left(\frac{1}{2}\times(3+4)\times0.2+\frac{1}{2}\times(3+4)\times0.6\right)/7\times(0.2+0.6)\right)^4\\ -\left(\left(\frac{1}{2}\times(1+2)\times0.4+\frac{1}{2}\times(2+4)\times0.6\right)/7\times(0.4+0.6)\right)^4\\ -\left(\left(\frac{1}{2}\times(5+6)\times0.3+\frac{1}{2}\times(4+5)\times0.7\right)/7\times(0.3+0.7)\right)^4\end{array}\right)} = s_{0.450}$$

$$Sc(\tilde{a}_2(p)) = s_{\frac{1}{2}\left(\begin{array}{c}1+\left(\left(\frac{1}{2}\times(2+3)\times0.4+\frac{1}{2}\times(3+4)\times0.2+\frac{1}{2}\times(5+6)\times0.2\right)/7\times(0.4+0.2+0.2)\right)^4\\ -\left(\left(\frac{1}{2}\times(4+5)\times0.3+\frac{1}{2}\times(2+3)\times0.6\right)/7\times(0.3+0.6)\right)^4\\ -\left(\left(\frac{1}{2}\times(1+2)\times0.3+\frac{1}{2}\times(2+3)\times0.4\right)/7\times(0.3+0.4)\right)^4\end{array}\right)} = s_{0.506}$$

We have $Sc(\tilde{a}_1(p)) \prec Sc(\tilde{a}_2(p))$, according to the Definition 11, then there is $\tilde{a}_1(p) \prec \tilde{a}_2(p)$.

Definition 12. Assume that $S_{[0,k-1]}$ is an LTS, for any two PULTSFNs $\tilde{a}_1(p) = \langle\left\{\left[s_{\mu^L_{1(t)}}, s_{\mu^U_{1(t)}}\right]\middle|\hat{p}_{1(t)}\right\}, \left\{\left[s_{\eta^L_{1(r)}}, s_{\eta^U_{1(r)}}\right]\middle|\tilde{p}_{1(r)}\right\}, \left\{\left[s_{\nu^L_{1(w)}}, s_{\nu^U_{1(w)}}\right]\middle|\bar{p}_{1(w)}\right\}\rangle$ and $\tilde{a}_2(p) = \langle\left\{\left[s_{\mu^L_{2(t)}}, s_{\mu^U_{2(t)}}\right]\middle|\hat{p}_{2(t)}\right\}, \left\{\left[s_{\eta^L_{2(r)}}, s_{\eta^U_{2(r)}}\right]\middle|\tilde{p}_{2(r)}\right\}, \left\{\left[s_{\nu^L_{2(w)}}, s_{\nu^U_{2(w)}}\right]\middle|\bar{p}_{2(w)}\right\}\rangle$, where $\#T_1=\#T_2=\#T$, $\#R_1=\#R_2=\#R$, $\#W_1=\#W_2=\#W$, and for the part of ULVs, function I (·) satisfies $I(s_\alpha) = \alpha$, then the Hamming distance $D_H(\tilde{a}_1(p), \tilde{a}_2(p))$ between $\tilde{a}_1(p)$ and $\tilde{a}_2(p)$ is defined as :

$$D_H(\tilde{a}_1(p), \tilde{a}_2(p)) = \frac{1}{6}\left(\begin{array}{c}\dfrac{\sum_{t=1}^{\#T}\left(\begin{array}{c}\left|\left(I(s_{\mu^L_{1(t)}})\hat{p}_{1(t)}\right)^q - \left(I(s_{\mu^L_{2(t)}})\hat{p}_{2(t)}\right)^q\right|\\ +\left|\left(I(s_{\mu^U_{1(t)}})\hat{p}_{1(t)}\right)^q - \left(I(s_{\mu^U_{2(t)}})\hat{p}_{2(t)}\right)^q\right|\end{array}\right)}{\#T} +\\ \dfrac{\sum_{r=1}^{\#R}\left(\begin{array}{c}\left|\left(I(s_{\eta^L_{1(r)}})\tilde{p}_{1(r)}\right)^q - \left(I(s_{\eta^L_{2(r)}})\tilde{p}_{2(r)}\right)^q\right|\\ +\left|\left(I(s_{\eta^U_{1(r)}})\tilde{p}_{1(r)}\right)^q - \left(I(s_{\eta^U_{2(r)}})\tilde{p}_{2(r)}\right)^q\right|\end{array}\right)}{\#R} +\\ \dfrac{\sum_{w=1}^{\#W}\left(\begin{array}{c}\left|\left(I(s_{\nu^L_{1(w)}})\bar{p}_{1(w)}\right)^q - \left(I(s_{\nu^L_{2(w)}})\bar{p}_{2(w)}\right)^q\right|\\ +\left|\left(I(s_{\nu^U_{1(w)}})\bar{p}_{1(w)}\right)^q - \left(I(s_{\nu^U_{2(w)}})\bar{p}_{2(w)}\right)^q\right|\end{array}\right)}{\#W}\end{array}\right) \quad (9)$$

Example 4. Taking two PULTSFNs $\tilde{a}_j(p^N)$(j=1, 2) in Example 2 as an example, using the Eq. (9) to calculate the Hamming distance between them, q = 4, then we have:

$$D_H(\tilde{a}_1(p),\tilde{a}_2(p)) = \frac{1}{6}\left(\begin{array}{c} \frac{1}{3}\left(\begin{array}{c} |(3\times 0.0)^4-(2\times 0.5)^4| + |(4\times 0.0)^4-(3\times 0.5)^4| + \\ |(3\times 0.25)^4-(3\times 0.25)^4| + |(4\times 0.25)^4-(4\times 0.25)^4| + \\ |(4\times 0.75)^4-(5\times 0.25)^4| + |(5\times 0.75)^4-(6\times 0.25)^4| \end{array}\right) \\ +\frac{1}{3}\left(\begin{array}{c} |(1\times 0.4)^4-(2\times 0.0)^4| + |(2\times 0.4)^4-(3\times 0.0)^4| + \\ |(2\times 0.3)^4-(2\times 0.67)^4| + |(3\times 0.3)^4-(3\times 0.67)^4| + \\ |(3\times 0.3)^4-(4\times 0.33)^4| + |(4\times 0.3)^4-(5\times 0.33)^4| \end{array}\right) \\ +\frac{1}{2}\left(\begin{array}{c} |(4\times 0.7)^4-(1\times 0.43)^4| + |(5\times 0.7)^4-(2\times 0.43)^4| + \\ |(5\times 0.3)^4-(2\times 0.57)^4| + |(6\times 0.3)^4-(3\times 0.57)^4| \end{array}\right) \end{array}\right)$$
$$= 34.924$$

4. Some AOs for PULTSFS

The AO is a mathematical function that aggregates or fuses multiple values into a single value. The aggregation results provide important decision-making basis for dealing with MCDM problems. Therefore, some basic operation rules of PULTSFNs in this section are defined. On this basis, some AOs are developed, namely PULTSFWA, PULTSFOWA, PULTSFWG and PULTSFOWG operators, which are used to fuse the evaluation information of DMs.

4.1 The basic operational rules of PULTSFNs

Definition 13. Assume that $S_{[0,k-1]}$ is a LTS, for any three PULTSFNs, $\tilde{a}(p) = \left\langle\left\{\left[s_{\mu^L_{(t)}}, s_{\mu^U_{(t)}}\right]\middle|\hat{p}_{(t)}\right\}, \left\{\left[s_{\eta^L_{(r)}}, s_{\eta^U_{(r)}}\right]\middle|\tilde{p}_{(r)}\right\}, \left\{\left[s_{\nu^L_{(w)}}, s_{\nu^U_{(w)}}\right]\middle|\bar{p}_{(w)}\right\}\right\rangle$, $\tilde{a}_1(p) = \left\langle\left\{\left[s_{\mu^L_{1(t)}}, s_{\mu^U_{1(t)}}\right]\middle|\hat{p}_{1(t)}\right\}, \left\{\left[s_{\eta^L_{1(r)}}, s_{\eta^U_{1(r)}}\right]\middle|\tilde{p}_{1(r)}\right\}, \left\{\left[s_{\nu^L_{1(w)}}, s_{\nu^U_{1(w)}}\right]\middle|\bar{p}_{1(w)}\right\}\right\rangle$, and $\tilde{a}_2(p) = \left\langle\left\{\left[s_{\mu^L_{2(t)}}, s_{\mu^U_{2(t)}}\right]\middle|\hat{p}_{2(t)}\right\}, \left\{\left[s_{\eta^L_{2(r)}}, s_{\eta^U_{2(r)}}\right]\middle|\tilde{p}_{2(r)}\right\}, \left\{\left[s_{\nu^L_{2(w)}}, s_{\nu^U_{2(w)}}\right]\middle|\bar{p}_{2(w)}\right\}\right\rangle$, the parameter λ>0, then the basic operation rules of these PULTSFNs are as follows :

(1) $(\tilde{a}(p))^c = \left\langle\left\{\left[s_{\nu^L_{(w)}}, s_{\nu^U_{(w)}}\right]\middle|\bar{p}_{(w)}\right\}, \left\{\left[s_{\eta^L_{(r)}}, s_{\eta^U_{(r)}}\right]\middle|\tilde{p}_{(r)}\right\}, \left\{\left[s_{\mu^L_{(t)}}, s_{\mu^U_{(t)}}\right]\middle|\hat{p}_{(t)}\right\}\right\rangle$;

(2) $$\tilde{a}_1(p)\oplus\tilde{a}_2(p) = \left(\begin{array}{c} \left\{\left[s_{k\sqrt[q]{1-\prod_{i=1}^{2}\left(1-\left(\mu^L_{i(t)}/k\right)^q\right)}}, s_{k\sqrt[q]{1-\prod_{i=1}^{2}\left(1-\left(\mu^U_{i(t)}/k\right)^q\right)}}\right]\middle|\frac{1}{2}\sum_{i=1}^{2}\hat{p}_{i(t)}\right\}, \\ \left\{\left[s_{\prod_{i=1}^{2}\left(\eta^L_{i(r)}/k\right)}, s_{\prod_{i=1}^{2}\left(\eta^U_{i(r)}/k\right)}\right]\middle|\left(\prod_{i=1}^{2}\tilde{p}_{i(r)}\right)^{1/2}\right\}, \\ \left\{\left[s_{\prod_{i=1}^{2}\left(\nu^L_{i(w)}/k\right)}, s_{\prod_{i=1}^{2}\left(\nu^U_{i(w)}/k\right)}\right]\middle|\left(\prod_{i=1}^{2}\bar{p}_{i(w)}\right)^{1/2}\right\} \end{array}\right);$$

(3) $$\tilde{a}_1(p)\otimes\tilde{a}_2(p) = \left(\begin{array}{c} \left\{\left[s_{\prod_{i=1}^{2}\left(\mu^L_{i(t)}/k\right)}, s_{\prod_{i=1}^{2}\left(\mu^U_{i(t)}/k\right)}\right]\middle|\left(\prod_{i=1}^{2}\hat{p}_{i(t)}\right)^{1/2}\right\}, \\ \left\{\left[s_{k\sqrt[q]{1-\prod_{i=1}^{2}\left(1-\left(\eta^L_{i(r)}/k\right)^q\right)}}, s_{k\sqrt[q]{1-\prod_{i=1}^{2}\left(1-\left(\eta^U_{i(r)}/k\right)^q\right)}}\right]\middle|\frac{1}{2}\sum_{i=1}^{2}\tilde{p}_{i(r)}\right\}, \\ \left\{\left[s_{k\sqrt[q]{1-\prod_{i=1}^{2}\left(1-\left(\nu^L_{i(w)}/k\right)^q\right)}}, s_{k\sqrt[q]{1-\prod_{i=1}^{2}\left(1-\left(\nu^U_{i(w)}/k\right)^q\right)}}\right]\middle|\frac{1}{2}\sum_{i=1}^{2}\bar{p}_{i(w)}\right\} \end{array}\right);$$

$$(4)\ \lambda(\tilde{a}(p))=\left(\begin{array}{c}\left\{\left[s_{k\sqrt[q]{1-\left(1-\left(\mu^{L}_{(t)}/k\right)^{q}\right)^{\lambda}}},s_{k\sqrt[q]{1-\left(1-\left(\mu^{U}_{(t)}/k\right)^{q}\right)^{\lambda}}}\right]\left|\hat{p}_{(t)}\right.\right\},\\ \left\{\left[s_{k\left(\eta^{L}_{(r)}/k\right)^{\lambda}},s_{k\left(\eta^{U}_{(r)}/k\right)^{\lambda}}\right]\left|\tilde{p}_{(r)}\right.\right\},\left\{\left[s_{k\left(\nu^{L}_{(w)}/k\right)^{\lambda}},s_{k\left(\nu^{U}_{(w)}/k\right)^{\lambda}}\right]\left|\bar{p}_{(w)}\right.\right\}\end{array}\right);$$

$$(5)\ (\tilde{a}(p))^{\lambda}=\left\{\begin{array}{c}\left\{\left[s_{k\left(\mu^{L}_{(t)}/k\right)^{\lambda}},s_{k\left(\mu^{U}_{(t)}/k\right)^{\lambda}}\right]\left|\hat{p}_{(t)}\right.\right\},\left\{\left[\begin{array}{c}s_{k\sqrt[q]{1-\left(1-\left(\eta^{L}_{(r)}/k\right)^{q}\right)^{\lambda}}},\\ s_{k\sqrt[q]{1-\left(1-\left(\eta^{U}_{(r)}/k\right)^{q}\right)^{\lambda}}}\end{array}\right]\left|\tilde{p}_{(r)}\right.\right\},\\ \left\{\left[s_{k\sqrt[q]{1-\left(1-\left(\nu^{L}_{(w)}/k\right)^{q}\right)^{\lambda}}},s_{k\sqrt[q]{1-\left(1-\left(\nu^{U}_{(w)}/k\right)^{q}\right)^{\lambda}}}\right]\left|\bar{p}_{(w)}\right.\right\}\end{array}\right\}.$$

Theorem 1. Suppose that for any three PULTSFNs, $\tilde{a}(p)=\langle\left\{\left[s_{\mu^{L}_{(t)}},s_{\mu^{U}_{(t)}}\right]\left|\hat{p}_{(t)}\right.\right\},\left\{\left[s_{\eta^{L}_{(r)}},s_{\eta^{U}_{(r)}}\right]\left|\tilde{p}_{(r)}\right.\right\},\left\{\left[s_{\nu^{L}_{(w)}},s_{\nu^{U}_{(w)}}\right]\left|\bar{p}_{(w)}\right.\right\}\rangle$, $\tilde{a}_1(p)=\left\langle\left\{\left[s_{\mu^{L}_{1(t)}},s_{\mu^{U}_{1(t)}}\right]\left|\hat{p}_{1(t)}\right.\right\},\left\{\left[s_{\eta^{L}_{1(r)}},s_{\eta^{U}_{1(r)}}\right]\left|\tilde{p}_{1(r)}\right.\right\},\left\{\left[s_{\nu^{L}_{1(w)}},s_{\nu^{U}_{1(w)}}\right]\left|\bar{p}_{1(w)}\right.\right\}\right\rangle$, and $\tilde{a}_2(p)=\langle\left\{\left[s_{\mu^{L}_{2(t)}},s_{\mu^{U}_{2(t)}}\right]\left|\hat{p}_{2(t)}\right.\right\},\left\{\left[s_{\eta^{L}_{2(r)}},s_{\eta^{U}_{2(r)}}\right]\left|\tilde{p}_{2(r)}\right.\right\},\left\{\left[s_{\nu^{L}_{2(w)}},s_{\nu^{U}_{2(w)}}\right]\left|\bar{p}_{2(w)}\right.\right\}\rangle$, the parameters λ, λ_1, $\lambda_2>0$, then

(1)$\tilde{a}_1(p)\oplus\tilde{a}_2(p)=\tilde{a}_2(p)\oplus\tilde{a}_1(p)$;

(2)$\tilde{a}_1(p)\otimes\tilde{a}_2(p)=\tilde{a}_2(p)\otimes\tilde{a}_1(p)$;

(3)$\lambda(\tilde{a}_1(p)\oplus\tilde{a}_2(p))=\lambda\tilde{a}_1(p)\oplus\lambda\tilde{a}_2(p)$;

(4)$\lambda_1\tilde{a}(p)\oplus\lambda_2\tilde{a}(p)=(\lambda_1+\lambda_2)\tilde{a}(p)$;

(5)$(\tilde{a}(p))^{\lambda_1}\otimes(\tilde{a}(p))^{\lambda_2}=(\tilde{a}(p))^{\lambda_1+\lambda_2}$;

(6)$(\tilde{a}_1(p))^{\lambda}\otimes(\tilde{a}_2(p))^{\lambda}=(\tilde{a}_1(p)\otimes\tilde{a}_2(p))^{\lambda}$.

The above optional properties can be easily proved according to Definition 13.

Based on the above optional rules of PULTSFNs, we develop the following series of PULTSF AOs.

4.2 PULTSFWA and PULTSFOWA operators

Definition 14. Let there be a set of PULTSFNs $\tilde{a}_i(p)=\left\langle\left\{\left[s_{\mu^{L}_{i(t)}},s_{\mu^{U}_{i(t)}}\right]\left|\hat{p}_{i(t)}\right.\right\},\left\{\left[s_{\eta^{L}_{i(r)}},s_{\eta^{U}_{i(r)}}\right]\left|\tilde{p}_{i(r)}\right.\right\},\left\{\left[s_{\nu^{L}_{i(w)}},s_{\nu^{U}_{i(w)}}\right]\left|\bar{p}_{i(w)}\right.\right\}\right\rangle$ (i=1, 2,..., n), and the corresponding weight vector is $\boldsymbol{\omega}=(\omega_1, \omega_2, ..., \omega_n)^T$, satisfying $\omega_i>0$ and $\sum_{i=1}^{n}\omega_i=1$. Then the PULTSFWA operator is a mapping $\Theta^n\rightarrow\Theta$ such that:

$$PULTSFWA(\tilde{a}_1(p),\tilde{a}_2(p),\ldots,\tilde{a}_n(p))=\oplus_{i=1}^{n}(\omega_i\tilde{a}_i(p)) \tag{10}$$

Theorem 2. Let $\tilde{a}_i(p)=\langle\left\{\left[s_{\mu^{L}_{i(t)}},s_{\mu^{U}_{i(t)}}\right]\left|\hat{p}_{i(t)}\right.\right\},\left\{\left[s_{\eta^{L}_{i(r)}},s_{\eta^{U}_{i(r)}}\right]\left|\tilde{p}_{i(r)}\right.\right\},\left\{\left[s_{\nu^{L}_{i(w)}},s_{\nu^{U}_{i(w)}}\right]\left|\bar{p}_{i(w)}\right.\right\}\rangle$ (i=1, 2,..., n) be the PULTSF array, then the result obtained by the PULTSFWA operator aggregation is still PULTSFN. The AO is expressed as:

$$PULTSFWA(\tilde{a}_1(p), \tilde{a}_2(p), \ldots, \tilde{a}_n(p)) = \left(\begin{array}{c} \left\{ \left[s_{k\sqrt[q]{1-\prod_{i=1}^{n}\left(1-\left(\mu_{i(t)}^{L}/k\right)^{q}\right)^{\omega_i}}}, s_{k\sqrt[q]{1-\prod_{i=1}^{n}\left(1-\left(\mu_{i(t)}^{U}/k\right)^{q}\right)^{\omega_i}}} \right] \middle| \sum_{i=1}^{n} \omega_i \hat{p}_{i(t)} \right\}, \\ \left\{ \left[s_{k\prod_{i=1}^{n}\left(\eta_{i(r)}^{L}/k\right)^{\omega_i}}, s_{k\prod_{i=1}^{n}\left(\eta_{i(r)}^{U}/k\right)^{\omega_i}} \right] \middle| \prod_{i=1}^{n} \left(\tilde{p}_{i(r)}\right)^{\omega_i} \right\}, \\ \left\{ \left[s_{k\prod_{i=1}^{n}\left(\nu_{i(w)}^{L}/k\right)^{\omega_i}}, s_{k\prod_{i=1}^{n}\left(\nu_{i(w)}^{U}/k\right)^{\omega_i}} \right] \middle| \prod_{i=1}^{n} \left(\bar{p}_{i(w)}\right)^{\omega_i} \right\} \end{array} \right) \tag{11}$$

Proof : Based on the operational rules in Definition 13, the Theorem 2 can be proved by mathematical induction. For *n*= 2, suppose $\tilde{a}_i(p)$ (*i*=1, 2) are two PULTSFNs, then by Definition 13, we have:

$$\omega_i(\tilde{a}_i(p)) = \left(\begin{array}{c} \left\{ \left[s_{k\sqrt[q]{1-\left(1-\left(\mu_{i(t)}^{L}/k\right)^{q}\right)^{\omega_i}}}, s_{k\sqrt[q]{1-\left(1-\left(\mu_{i(t)}^{U}/k\right)^{q}\right)^{\omega_i}}} \right] \middle| \omega_i \hat{p}_{i(t)} \right\}, \\ \left\{ \left[s_{k\left(\eta_{i(r)}^{L}/k\right)^{\omega_i}}, s_{k\left(\eta_{i(r)}^{U}/k\right)^{\omega_i}} \right] \middle| \left(\tilde{p}_{i(r)}\right)^{\omega_i} \right\}, \left\{ \left[s_{k\left(\nu_{i(w)}^{L}/k\right)^{\omega_i}}, s_{k\left(\nu_{i(w)}^{U}/k\right)^{\omega_i}} \right] \middle| \left(\bar{p}_{i(w)}\right)^{\omega_i} \right\} \end{array} \right)$$

Then,

$$PULTSFWA(\tilde{a}_1(p), \tilde{a}_2(p)) = \omega_1(\tilde{a}_1(p)) \oplus \omega_2(\tilde{a}_2(p))$$

$$= \left(\begin{array}{c} \left\{ \left[s_{k\sqrt[q]{1-\left(1-\left(\mu_{1(t)}^{L}/k\right)^{q}\right)^{\omega_1}}}, s_{k\sqrt[q]{1-\left(1-\left(\mu_{1(t)}^{U}/k\right)^{q}\right)^{\omega_1}}} \right] \middle| \omega_1 \hat{p}_{1(t)} \right\}, \\ \left\{ \left[s_{k\left(\eta_{1(r)}^{L}/k\right)^{\omega_1}}, s_{k\left(\eta_{1(r)}^{U}/k\right)^{\omega_1}} \right] \middle| \left(\tilde{p}_{1(r)}\right)^{\omega_1} \right\}, \left\{ \left[s_{k\left(\nu_{1(w)}^{L}/k\right)^{\omega_1}}, s_{k\left(\nu_{1(w)}^{U}/k\right)^{\omega_1}} \right] \middle| \left(\bar{p}_{1(w)}\right)^{\omega_1} \right\} \end{array} \right)$$

$$\oplus \left(\begin{array}{c} \left\{ \left[s_{k\sqrt[q]{1-\left(1-\left(\mu_{2(t)}^{L}/k\right)^{q}\right)^{\omega_2}}}, s_{k\sqrt[q]{1-\left(1-\left(\mu_{2(t)}^{U}/k\right)^{q}\right)^{\omega_2}}} \right] \middle| \omega_2 \hat{p}_{2(t)} \right\}, \\ \left\{ \left[s_{k\left(\eta_{2(r)}^{L}/k\right)^{\omega_2}}, s_{k\left(\eta_{2(r)}^{U}/k\right)^{\omega_2}} \right] \middle| \left(\tilde{p}_{2(r)}\right)^{\omega_2} \right\}, \left\{ \left[s_{k\left(\nu_{2(w)}^{L}/k\right)^{\omega_2}}, s_{k\left(\nu_{2(w)}^{U}/k\right)^{\omega_2}} \right] \middle| \left(\bar{p}_{2(w)}\right)^{\omega_2} \right\} \end{array} \right)$$

$$= \left(\begin{array}{c} \left\{ \left[s_{k\sqrt[q]{1-\prod_{i=1}^{2}\left(1-\left(\mu_{i(t)}^{L}/k\right)^{q}\right)^{\omega_i}}}, s_{k\sqrt[q]{1-\prod_{i=1}^{2}\left(1-\left(\mu_{i(t)}^{U}/k\right)^{q}\right)^{\omega_i}}} \right] \middle| \sum_{i=1}^{2} \omega_i \hat{p}_{i(t)} \right\}, \\ \left\{ \left[s_{k\prod_{i=1}^{2}\left(\eta_{i(r)}^{L}/k\right)^{\omega_i}}, s_{k\prod_{i=1}^{2}\left(\eta_{i(r)}^{U}/k\right)^{\omega_i}} \right] \middle| \prod_{i=1}^{2} \left(\tilde{p}_{i(r)}\right)^{\omega_i} \right\}, \\ \left\{ \left[s_{k\prod_{i=1}^{2}\left(\nu_{i(w)}^{L}/k\right)^{\omega_i}}, s_{k\prod_{i=1}^{2}\left(\nu_{i(w)}^{U}/k\right)^{\omega_i}} \right] \middle| \prod_{i=1}^{2} \left(\bar{p}_{i(w)}\right)^{\omega_i} \right\} \end{array} \right)$$

Therefore, this result is also valid for *n* = 2.

Assume that the result is valid for *n*=*ϑ*, then Eq.(11) holds :

$$PULTSFWA(\tilde{a}_1(p),\tilde{a}_2(p),\ldots,\tilde{a}_\theta(p)) = \left(\begin{array}{c} \left\{\left[s_{k\sqrt[q]{1-\prod_{i=1}^{\theta}\left(1-\left(\mu_{i(t)}^{L}/k\right)^{q}\right)^{\omega_i}}}, s_{k\sqrt[q]{1-\prod_{i=1}^{\theta}\left(1-\left(\mu_{i(t)}^{U}/k\right)^{q}\right)^{\omega_i}}}\right]\left|\sum_{i=1}^{\theta}\omega_i\,\hat{p}_{i(t)}\right.\right\}, \\ \left\{\left[s_{k\prod_{i=1}^{\theta}\left(\eta_{i(r)}^{L}/k\right)^{\omega_i}}, s_{k\prod_{i=1}^{\theta}\left(\eta_{i(r)}^{U}/k\right)^{\omega_i}}\right]\left|\prod_{i=1}^{\theta}\left(\tilde{p}_{i(r)}\right)^{\omega_i}\right.\right\}, \\ \left\{\left[s_{k\prod_{i=1}^{\theta}\left(\nu_{i(w)}^{L}/k\right)^{\omega_i}}, s_{k\prod_{i=1}^{\theta}\left(\nu_{i(w)}^{U}/k\right)^{\omega_i}}\right]\left|\prod_{i=1}^{\theta}\left(\bar{p}_{i(w)}\right)^{\omega_i}\right.\right\} \end{array}\right)$$

Then, when n=ϑ+1, according to the algorithm of Definition 13, we can obtain:

$$PULTSFWA(\tilde{a}_1(p),\tilde{a}_2(p),\ldots,\tilde{a}_\theta(p),\tilde{a}_{\theta+1}(p))$$

$$= \left(\begin{array}{c} \left\{\left[s_{k\sqrt[q]{1-\prod_{i=1}^{\theta}\left(1-\left(\mu_{i(t)}^{L}/k\right)^{q}\right)^{\omega_i}}}, s_{k\sqrt[q]{1-\prod_{i=1}^{\theta}\left(1-\left(\mu_{i(t)}^{U}/k\right)^{q}\right)^{\omega_i}}}\right]\left|\sum_{i=1}^{\theta}\omega_i\,\hat{p}_{i(t)}\right.\right\}, \\ \left\{\left[s_{k\prod_{i=1}^{\theta}\left(\eta_{i(r)}^{L}/k\right)^{\omega_i}}, s_{k\prod_{i=1}^{\theta}\left(\eta_{i(r)}^{U}/k\right)^{\omega_i}}\right]\left|\prod_{i=1}^{\theta}\left(\tilde{p}_{i(r)}\right)^{\omega_i}\right.\right\}, \\ \left\{\left[s_{k\prod_{i=1}^{\theta}\left(\nu_{i(w)}^{L}/k\right)^{\omega_i}}, s_{k\prod_{i=1}^{\theta}\left(\nu_{i(w)}^{U}/k\right)^{\omega_i}}\right]\left|\prod_{i=1}^{\theta}\left(\bar{p}_{i(w)}\right)^{\omega_i}\right.\right\} \end{array}\right)$$

$$\oplus \left(\begin{array}{c} \left\{\left[s_{k\sqrt[q]{1-\left(1-\left(\mu_{\theta+1(t)}^{L}/k\right)^{q}\right)^{\omega_{\theta+1}}}}, s_{k\sqrt[q]{1-\left(1-\left(\mu_{\theta+1(t)}^{U}/k\right)^{q}\right)^{\omega_{\theta+1}}}}\right]\left|\omega_{\theta+1}\hat{p}_{\theta+1(t)}\right.\right\}, \\ \left\{\left[s_{k\left(\eta_{\theta+1(r)}^{L}/k\right)^{\omega_{\theta+1}}}, s_{k\left(\eta_{\theta+1(r)}^{U}/k\right)^{\omega_{\theta+1}}}\right]\left|\left(\tilde{p}_{\theta+1(r)}\right)^{\omega_{\theta+1}}\right.\right\}, \\ \left\{\left[s_{k\left(\nu_{\theta+1(w)}^{L}/k\right)^{\omega_{\theta+1}}}, s_{k\left(\nu_{\theta+1(w)}^{U}/k\right)^{\omega_{\theta+1}}}\right]\left|\left(\bar{p}_{\theta+1(w)}\right)^{\omega_{\theta+1}}\right.\right\} \end{array}\right)$$

$$= \left(\begin{array}{c} \left\{\left[s_{k\sqrt[q]{1-\prod_{i=1}^{\theta+1}\left(1-\left(\mu_{i(t)}^{L}/k\right)^{q}\right)^{\omega_i}}}, s_{k\sqrt[q]{1-\prod_{i=1}^{\theta+1}\left(1-\left(\mu_{i(t)}^{U}/k\right)^{q}\right)^{\omega_i}}}\right]\left|\sum_{i=1}^{\theta+1}\omega_i\,\hat{p}_{i(t)}\right.\right\}, \\ \left\{\left[s_{k\prod_{i=1}^{\theta+1}\left(\eta_{i(r)}^{L}/k\right)^{\omega_i}}, s_{k\prod_{i=1}^{\theta+1}\left(\eta_{i(r)}^{U}/k\right)^{\omega_i}}\right]\left|\prod_{i=1}^{\theta+1}\left(\tilde{p}_{i(r)}\right)^{\omega_i}\right.\right\}, \\ \left\{\left[s_{k\prod_{i=1}^{\theta+1}\left(\nu_{i(w)}^{L}/k\right)^{\omega_i}}, s_{k\prod_{i=1}^{\theta+1}\left(\nu_{i(w)}^{U}/k\right)^{\omega_i}}\right]\left|\prod_{i=1}^{\theta+1}\left(\bar{p}_{i(w)}\right)^{\omega_i}\right.\right\} \end{array}\right)$$

Obviously, the equation is also valid for n=ϑ+1, so the Eq.(11) is valid for all n.

The following is to explore the basic properties of PULTSFWA operator :

Theorem 3. (Monotonicity) Let $\tilde{a}_i(p)$ and $\tilde{a}_i'(p) = \langle\left\{\left[s_{\mu_{i(t)}'^{L}}, s_{\mu_{i(t)}'^{U}}\right]\left|\widehat{p'}_{i(t)}\right.\right\}, \left\{\left[s_{\eta_{i(r)}'^{L}}, s_{\eta_{i(r)}'^{U}}\right]\left|\widetilde{p'}_{i(r)}\right.\right\}, \left\{\left[s_{\nu_{i(w)}'^{L}}, s_{\nu_{i(w)}'^{U}}\right]\left|\overline{p'}_{i(w)}\right.\right\}\rangle$ (i=1, 2,..., n) be two groups of PULTSFNs, if for all i, $\mu_{i(t)}^{U}\hat{p}_{i(t)} \le \mu_{i(t)}'^{U}\widehat{p'}_{i(t)}$, $\eta_{i(r)}^{U}\tilde{p}_{i(r)} \ge \eta_{i(r)}'^{U}\widetilde{p'}_{i(r)}$ and $\nu_{i(w)}^{U}\bar{p}_{i(w)} \ge \nu_{i(w)}'^{U}\overline{p'}_{i(w)}$, then:

$$PULTSFWA(\tilde{a}_1(p),\tilde{a}_2(p),\ldots,\tilde{a}_n(p)) \le PULTSFWA(\tilde{a}_1'(p),\tilde{a}_2'(p),\ldots,\tilde{a}_n'(p)) \quad (12)$$

Theorem 4. (idempotency) If $\tilde{a}_i(p)$ (i=1, 2,..., n) are equal, i.e. $\tilde{a}_i(p) = \tilde{a}(p) = \langle\left\{\left[s_{\mu_{(t)}^{L}}, s_{\mu_{(t)}^{U}}\right]\left|\hat{p}_{(t)}\right.\right\}, \left\{\left[s_{\eta_{(r)}^{L}}, s_{\eta_{(r)}^{U}}\right]\left|\tilde{p}_{(r)}\right.\right\}, \left\{\left[s_{\nu_{(w)}^{L}}, s_{\nu_{(w)}^{U}}\right]\left|\bar{p}_{(w)}\right.\right\}\rangle$($i$=1, 2,..., n), then

$$PULTSFWA(\tilde{a}_1(p),\tilde{a}_2(p),\dots,\tilde{a}_n(p))=\tilde{a}(p) \quad (13)$$

Theorem 5. (boundedness) Let $\tilde{a}_i(p)$ (i=1, 2,..., n) is a set of PULTSFNs, then

$$\tilde{d}^- \le PULTSFWA(\tilde{a}_1(p),\tilde{a}_2(p),\dots,\tilde{a}_n(p)) \le \tilde{d}^+ \quad (14)$$

where $\tilde{d}^+$ is the PULTSFN corresponding to $\max_i Sc(\tilde{a}_i(p))$ and $\tilde{d}^-$ is the PULTSFN corresponding to $\min_i Sc(\tilde{a}_i(p))$.

Definition 15. Let $\tilde{a}_i(p)=\langle\left\{\left[s_{\mu^L_{i(t)}},s_{\mu^U_{i(t)}}\right]\middle|\hat{p}_{i(t)}\right\},\left\{\left[s_{\eta^L_{i(r)}},s_{\eta^U_{i(r)}}\right]\middle|\tilde{p}_{i(r)}\right\},\left\{\left[s_{\nu^L_{i(w)}},s_{\nu^U_{i(w)}}\right]\middle|\bar{p}_{i(w)}\right\}\rangle$ (i=1, 2,..., n) be a set of PULTSF arrays. Suppose that PULTSFOWA :$\Theta^n \to \Theta$, if :

$$PULTSFOWA(\tilde{a}_1(p),\tilde{a}_2(p),\dots,\tilde{a}_n(p))=\oplus_{j=1}^{n}\left(\omega_j\tilde{a}_{\sigma(j)}(p)\right) \quad (15)$$

Then PULTSFOWA is called the probabilistic uncertain linguistic T-spherical fuzzy ordered weighted averaging operator, referred to as PULTSFOWA operator, where (σ(1), σ(2), ..., σ(n)) is a permutation of (1,2,...,n), and σ(j-1)≥σ(j) (j=1,2,...,n). $\tilde{a}_{\sigma(j)}(p)$ is the PULTSFN with the largest $\tilde{a}_i(p)$(i=1,2,...,n), and the corresponding weight ω_j satisfies $\omega_j \ge 0$ and $\sum_{j=1}^{n}\omega_j=1$.

Theorem 6. Let $\tilde{a}_i(p)=\langle\left\{\left[s_{\mu^L_{i(t)}},s_{\mu^U_{i(t)}}\right]\middle|\hat{p}_{i(t)}\right\},\left\{\left[s_{\eta^L_{i(r)}},s_{\eta^U_{i(r)}}\right]\middle|\tilde{p}_{i(r)}\right\},\left\{\left[s_{\nu^L_{i(w)}},s_{\nu^U_{i(w)}}\right]\middle|\bar{p}_{i(w)}\right\}\rangle$ (i=1, 2,..., n) be a set of PULTSF arrays, then:

$$PULTSFOWA(\tilde{a}_1(p),\tilde{a}_2(p),\dots,\tilde{a}_n(p))$$
$$=\left(\begin{array}{c}\left\{\left[s_{k\sqrt[q]{1-\prod_{j=1}^{n}\left(1-\left(\mu^L_{\sigma(j)(t)}/k\right)^q\right)^{\omega_j}}},s_{k\sqrt[q]{1-\prod_{j=1}^{n}\left(1-\left(\mu^U_{\sigma(j)(t)}/k\right)^q\right)^{\omega_j}}}\right]\middle|\sum_{j=1}^{n}\omega_j\hat{p}_{\sigma(j)(t)}\right\}, \\ \left\{\left[s_{k\prod_{j=1}^{n}\left(\eta^L_{\sigma(j)(r)}/k\right)^{\omega_j}},s_{k\prod_{j=1}^{n}\left(\eta^U_{\sigma(j)(r)}/k\right)^{\omega_j}}\right]\middle|\prod_{j=1}^{n}\left(\tilde{p}_{\sigma(j)(r)}\right)^{\omega_j}\right\}, \\ \left\{\left[s_{k\prod_{j=1}^{n}\left(\nu^L_{\sigma(j)(w)}/k\right)^{\omega_j}},s_{k\prod_{j=1}^{n}\left(\nu^U_{\sigma(j)(w)}/k\right)^{\omega_j}}\right]\middle|\prod_{j=1}^{n}\left(\bar{p}_{\sigma(j)(w)}\right)^{\omega_j}\right\}\end{array}\right) \quad (16)$$

Proof : Similar to the proof of Theorem 2.

The PULTSFOWA operator is easily proved to have the properties of idempotence, boundedness and monotonicity.

4.3 PULTSFWG and PULTSFOWG operators

Definition 16. Let $\tilde{a}_i(p)=\langle\left\{\left[s_{\mu^L_{i(t)}},s_{\mu^U_{i(t)}}\right]\middle|\hat{p}_{i(t)}\right\},\left\{\left[s_{\eta^L_{i(r)}},s_{\eta^U_{i(r)}}\right]\middle|\tilde{p}_{i(r)}\right\},\left\{\left[s_{\nu^L_{i(w)}},s_{\nu^U_{i(w)}}\right]\middle|\bar{p}_{i(w)}\right\}\rangle$ (i=1, 2,..., n) be a set of PULTSFNs, and the corresponding weight vector is $\boldsymbol{\omega}=(\omega_1,\omega_2,\dots,\omega_n)^T$, satisfying $\omega_i > 0$ and $\sum_{i=1}^{n}\omega_i=1$. Then the PULTSFWG operator is a mapping $\Theta^n \to \Theta$, satisfying

$$PULTSFWG(\tilde{a}_1(p),\tilde{a}_2(p),\dots,\tilde{a}_n(p))=\otimes_{i=1}^{n}(\tilde{a}_i(p))^{\omega_i} \quad (17)$$

Theorem 7. Let $\tilde{a}_i(p)=\langle\left\{\left[s_{\mu^L_{i(t)}},s_{\mu^U_{i(t)}}\right]\middle|\hat{p}_{i(t)}\right\},\left\{\left[s_{\eta^L_{i(r)}},s_{\eta^U_{i(r)}}\right]\middle|\tilde{p}_{i(r)}\right\},\left\{\left[s_{\nu^L_{i(w)}},s_{\nu^U_{i(w)}}\right]\middle|\bar{p}_{i(w)}\right\}\rangle$ (i=1, 2,..., n) be a PULTSF array, then the result of aggregation by PULTSFWG operator is still PULTSFN. The AO is expressed as:

$$PULTSFWG(\tilde{a}_1(p),\tilde{a}_2(p),\ldots,\tilde{a}_n(p)) = \left(\begin{array}{c} \left\{\left[s_{k\prod_{i=1}^{n}\left(\mu_{i(t)}^{L}/k\right)^{\omega_i}}, s_{k\prod_{i=1}^{n}\left(\mu_{i(t)}^{U}/k\right)^{\omega_i}}\right] \middle| \prod_{i=1}^{n}\left(\hat{p}_{i(t)}\right)^{\omega_i}\right\}, \\ \left\{\left[s_{k\sqrt[q]{1-\prod_{i=1}^{n}\left(1-\left(\eta_{i(r)}^{L}/k\right)^{q}\right)^{\omega_i}}}, s_{k\sqrt[q]{1-\prod_{i=1}^{n}\left(1-\left(\eta_{i(r)}^{U}/k\right)^{q}\right)^{\omega_i}}}\right] \middle| \sum_{i=1}^{n}\omega_i\,\tilde{p}_{i(r)}\right\}, \\ \left\{\left[s_{k\sqrt[q]{1-\prod_{i=1}^{n}\left(1-\left(\nu_{i(w)}^{L}/k\right)^{q}\right)^{\omega_i}}}, s_{k\sqrt[q]{1-\prod_{i=1}^{n}\left(1-\left(\nu_{i(w)}^{U}/k\right)^{q}\right)^{\omega_i}}}\right] \middle| \sum_{i=1}^{n}\omega_i\,\bar{p}_{i(w)}\right\} \end{array}\right) \quad (18)$$

The proof process of this operator is similar to that of PULTSFWA operator.

The following is to explore the basic properties of PULTSFWG operator :

Theorem 8. (Monotonicity) Suppose that $\tilde{a}_i(p)$ and $\tilde{a}'_i(p) = \langle\left\{\left[s_{\mu'^{L}_{i(t)}}, s_{\mu'^{U}_{i(t)}}\right] \middle| \widehat{p'}_{i(t)}\right\}, \left\{\left[s_{\eta'^{L}_{i(r)}}, s_{\eta'^{U}_{i(r)}}\right] \middle| \widetilde{p'}_{i(r)}\right\}, \left\{\left[s_{\nu'^{L}_{i(w)}}, s_{\nu'^{U}_{i(w)}}\right] \middle| \overline{p'}_{i(w)}\right\}\rangle$ (*i*=1, 2,..., *n*) are two groups of PULTSFNs, if for all *i*, $\mu^{U}_{i(t)}\hat{p}_{i(t)} \le \mu'^{U}_{i(t)}\widehat{p'}_{i(t)}$,$\eta^{U}_{i(r)}\tilde{p}_{i(r)} \ge \eta'^{U}_{i(r)}\widetilde{p'}_{i(r)}$ and $\nu^{U}_{i(w)}\bar{p}_{i(w)} \ge \nu'^{U}_{i(w)}\overline{p'}_{i(w)}$, then:

$$PULTSFWG(\tilde{a}_1(p),\tilde{a}_2(p),\ldots,\tilde{a}_n(p)) \le PULTSFWG(\tilde{a}'_1(p),\tilde{a}'_2(p),\ldots,\tilde{a}'_n(p)) \quad (19)$$

Theorem 9. (idempotency) If $\tilde{a}_i(p)$ (*i*=1, 2,..., *n*) are equal, i.e. $\tilde{a}_i(p) = \tilde{a}(p) = \langle\left\{\left[s_{\mu^{L}_{(t)}}, s_{\mu^{U}_{(t)}}\right] \middle| \hat{p}_{(t)}\right\}, \left\{\left[s_{\eta^{L}_{(r)}}, s_{\eta^{U}_{(r)}}\right] \middle| \tilde{p}_{(r)}\right\}, \left\{\left[s_{\nu^{L}_{(w)}}, s_{\nu^{U}_{(w)}}\right] \middle| \bar{p}_{(w)}\right\}\rangle$(*i*=1, 2,..., *n*), then:

$$PULTSFWG(\tilde{a}_1(p),\tilde{a}_2(p),\ldots,\tilde{a}_n(p)) = \tilde{a}(p) \quad (20)$$

Theorem 10. (Boundedness) Let $\tilde{a}_i(p)$ (*i*=1, 2,..., *n*) be a set of PULTSFNs, then:

$$\tilde{d}^{-} \le PULTSFWG(\tilde{a}_1(p),\tilde{a}_2(p),\ldots,\tilde{a}_n(p)) \le \tilde{d}^{+} \quad (21)$$

where $\tilde{d}^{+}$ is the PULTSFN corresponding to $\max_i Sc(\tilde{a}_i(p))$, and $\tilde{d}^{-}$ is the PULTSFN corresponding to $\min_i Sc(\tilde{a}_i(p))$.

Definition 17. Let $\tilde{a}_i(p) = \langle\left\{\left[s_{\mu^{L}_{i(t)}}, s_{\mu^{U}_{i(t)}}\right] \middle| \hat{p}_{i(t)}\right\}, \left\{\left[s_{\eta^{L}_{i(r)}}, s_{\eta^{U}_{i(r)}}\right] \middle| \tilde{p}_{i(r)}\right\}, \left\{\left[s_{\nu^{L}_{i(w)}}, s_{\nu^{U}_{i(w)}}\right] \middle| \bar{p}_{i(w)}\right\}\rangle$ (*i*=1, 2,..., *n*) be a set of PULTSF arrays. Suppose that PULTSFOWG : $\Theta^{n} \to \Theta$, if :

$$PULTSFOWG(\tilde{a}_1(p),\tilde{a}_2(p),\ldots,\tilde{a}_n(p)) = \otimes_{j=1}^{n}\left(\tilde{a}_{\sigma(j)}(p)\right)^{\omega_j} \quad (22)$$

Then PULTSFOWG is called a probabilistic uncertain linguistic T-spherical fuzzy ordered weighted geometric operator, abbreviated as PULTSFOWG operator, where (σ(1), σ(2), ..., σ(*n*)) is a permutation of (1,2,...,*n*), and σ(*j*-1)≥σ(*j*) (j=1,2,...,*n*). $\tilde{a}_{\sigma(j)}(p)$ is the PULTSFN with the largest *σ*(*j*) in $\tilde{a}_i(p)$(*i*=1,2,...,*n*), and the corresponding ω_j weights satisfy $\omega_j \ge 0$ and $\sum_{j=1}^{n}\omega_j = 1$.

Theorem 11. Let $\tilde{a}_i(p) = \langle\left\{\left[s_{\mu^{L}_{i(t)}}, s_{\mu^{U}_{i(t)}}\right] \middle| \hat{p}_{i(t)}\right\}, \left\{\left[s_{\eta^{L}_{i(r)}}, s_{\eta^{U}_{i(r)}}\right] \middle| \tilde{p}_{i(r)}\right\}, \left\{\left[s_{\nu^{L}_{i(w)}}, s_{\nu^{U}_{i(w)}}\right] \middle| \bar{p}_{i(w)}\right\}\rangle$ (*i*=1, 2,..., *n*) be a set of PULTSF arrays, then

$$PULTSFOWG(\tilde{a}_1(p),\tilde{a}_2(p),\ldots,\tilde{a}_n(p)) = \left(\begin{array}{c} \left\{\left[s_{k\prod_{j=1}^{n}\left(\mu^L_{\sigma(j)(t)}/k\right)^{\omega_j}}, s_{k\prod_{j=1}^{n}\left(\mu^U_{\sigma(j)(t)}/k\right)^{\omega_j}}\right] \middle| \prod_{j=1}^{n}\left(\hat{p}_{\sigma(j)(t)}\right)^{\omega_j}\right\}, \\ \left\{\left[s_{k\sqrt[q]{1-\prod_{j=1}^{n}\left(1-\left(\eta^L_{\sigma(j)(r)}/k\right)^q\right)^{\omega_j}}}, s_{k\sqrt[q]{1-\prod_{j=1}^{n}\left(1-\left(\eta^U_{\sigma(j)(r)}/k\right)^q\right)^{\omega_j}}}\right] \middle| \sum_{j=1}^{n}\omega_j\,\tilde{p}_{\sigma(j)(r)}\right\}, \\ \left\{\left[s_{k\sqrt[q]{1-\prod_{j=1}^{n}\left(1-\left(\nu^L_{\sigma(j)(w)}/k\right)^q\right)^{\omega_j}}}, s_{k\sqrt[q]{1-\prod_{j=1}^{n}\left(1-\left(\nu^U_{\sigma(j)(w)}/k\right)^q\right)^{\omega_j}}}\right] \middle| \sum_{j=1}^{n}\omega_j\,\bar{p}_{\sigma(j)(w)}\right\} \end{array} \right) \quad (23)$$

Proof : Similar to the proof of Theorem 2.

It is easy to prove that the PULTSFOWG operator has the properties of idempotence, boundedness and monotonicity.

5. Proposed algorithm of PULTSFS for MCDM problems

This section introduces a systematic decision-making method developed in the PULTSFS environment to solve the MCDM problem with completely unknown weights. In the general MCDM problem, decision makers will evaluate a limited set of alternatives : $\mathcal{H} = \{\hbar_i | i = 1,2,\ldots,m\}$. A related set of criteria: $\aleph = \{\beth_j | j = 1,2,\ldots,n\}$. Each criterion is assigned a relative importance represented by a weight vector : $\varpi = (\varpi_1,\varpi_2,\ldots,\varpi_n)^T$, where $\varpi_j \geq 0$ and $\sum_{j=1}^{n}\varpi_j = 1$. Experts use PULTSFNs to represent the evaluation information of each alternative under each criterion. These evaluation results are constructed into a decision matrix of PULTSF information. $\tilde{\mathcal{D}} = \left[\tilde{d}_{ij}(p)\right]_{m\times n}$, $\tilde{d}_{\text{ij}}(p) = \left\langle \left\{\left[s_{\mu^L_{ij(t)}}, s_{\mu^U_{ij(t)}}\right] | \hat{p}_{ij(t)}\right\}, \left\{\left[s_{\eta^L_{ij(r)}}, s_{\eta^U_{ij(r)}}\right] | \tilde{p}_{ij(r)}\right\} \quad \left\{\left[s_{\nu^L_{ij(w)}}, s_{\nu^U_{ij(w)}}\right] | \bar{p}_{ij(w)}\right\} \right\rangle$ (*i*=1, 2,..., *m*; *j*=1, 2,..., *n*; *t*=1, 2,..., #*T*; *r*=1, 2,..., #*R*; *w*=1, 2,..., #*W*).

Algorithm: MCDM within the PULTSFS setting

The MCDM proposed in the PULTSFS environment provides a systematic method for dealing with uncertain, hesitant and ambiguous complex information. The structure of the algorithm is designed to guide DMs from problem definition to the final ranking of alternatives. The whole process is described in detail as follows:

Step 1. Based on the initial PULTSF decision matrix $\tilde{\mathcal{D}}$, the PULTSFNs are normalized by Definition 9, and then the cost-type and benefit-type criteria values are converted by Eq.(24) to realize the standardization of criteria evaluation information in the PULTSF environment. Furthermore, the standardized PULTSF decision matrix $\tilde{\mathcal{G}} = \left[\tilde{\mathscr{g}}_{ij}(p)\right]_{m\times n}$ is obtained.

$$\tilde{\mathscr{g}}_{ij}(p) = \begin{cases} \tilde{d}_{ij}(p) = \left(\left\{\begin{bmatrix} s_{\mu^L_{ij(t)}}, \\ s_{\mu^U_{ij(t)}} \end{bmatrix} | \hat{p}_{ij(t)}\right\}, \left\{\begin{bmatrix} s_{\eta^L_{ij(r)}}, \\ s_{\eta^U_{ij(r)}} \end{bmatrix} | \tilde{p}_{ij(r)}\right\}, \left\{\begin{bmatrix} s_{\nu^L_{ij(w)}}, \\ s_{\nu^U_{ij(w)}} \end{bmatrix} | \bar{p}_{ij(w)}\right\}\right), j \in J_1 \\ \left(\tilde{d}_{ij}(p)\right)^c = \left(\left\{\begin{bmatrix} s_{\nu^L_{ij(w)}}, \\ s_{\nu^U_{ij(w)}} \end{bmatrix} | \bar{p}_{ij(w)}\right\}, \left\{\begin{bmatrix} s_{\eta^L_{ij(r)}}, \\ s_{\eta^U_{ij(r)}} \end{bmatrix} | \tilde{p}_{ij(r)}\right\}, \left\{\begin{bmatrix} s_{\mu^L_{ij(t)}}, \\ s_{\mu^U_{ij(t)}} \end{bmatrix} | \hat{p}_{ij(t)}\right\}\right), j \in J_2 \end{cases} \quad (24)$$

where J_1 and J_2 denote utility and cost criteria, respectively.

Step 2. Calculate the criteria weight value. The following is the steps of calculating the objective weight of criteria based on the standard deviation method of PULTSF environment :

Step 2.1. Use the PULTSFWA operator to calculate the average matrix $\mathcal{C} = \left[\tilde{c}_j(p)\right]_{1\times n}$, where the weights are equal.

$$\tilde{c}_j(p) = PULTSFWA\big(\tilde{\mathcal{g}}_{1j}(p), \tilde{\mathcal{g}}_{2j}(p), \ldots, \tilde{\mathcal{g}}_{mj}(p)\big) = \oplus_{i=1}^{m} \frac{1}{m} \tilde{\mathcal{g}}_{ij}(p)$$

$$= \left(\begin{array}{c} \left\{ \left[s_{k\sqrt[q]{1-\prod_{i=1}^{m}\left(1-\left(\mu_{ij(t)}^{L}/k\right)^{q}\right)^{1/m}}}, s_{k\sqrt[q]{1-\prod_{i=1}^{m}\left(1-\left(\mu_{ij(t)}^{U}/k\right)^{q}\right)^{1/m}}} \right] \middle| \sum_{i=1}^{m} \frac{1}{m} \hat{p}_{ij(t)} \right\}, \\ \left\{ \left[s_{k\prod_{i=1}^{m}\left(\eta_{ij(r)}^{L}/k\right)^{1/m}}, s_{k\prod_{i=1}^{m}\left(\eta_{ij(r)}^{U}/k\right)^{1/m}} \right] \middle| \prod_{i=1}^{m} \left(\tilde{p}_{ij(r)}\right)^{1/m} \right\}, \\ \left\{ \left[s_{k\prod_{i=1}^{m}\left(\nu_{ij(w)}^{L}/k\right)^{1/m}}, s_{k\prod_{i=1}^{m}\left(\nu_{ij(w)}^{U}/k\right)^{1/m}} \right] \middle| \prod_{i=1}^{m} \left(\bar{p}_{ij(w)}\right)^{1/m} \right\} \end{array} \right) \quad (25)$$

Step 2.2. The PULTSF Hamming distance is used to determine the standard deviation matrix $\mathcal{SD} = \left[sd_j\right]_{1\times n}$,

$$sd_j = \sqrt{\frac{1}{m}\sum_{i=1}^{m}\left(D_H\left(\tilde{\mathcal{g}}_{ij}(p), \tilde{c}_j(p)\right)\right)^2} \quad (26)$$

Step 2.3. Calculates the weight value of the criteria by using the Eq.(27) ϖ_j (j=1, 2,..., n),

$$\varpi_j = \frac{sd_j}{\sum_{j=1}^{n} sd_j} \quad (27)$$

Obviously, $\varpi_j \geq 0$ and $\sum_{j=1}^{n} \varpi_j = 1$.

Step 3. The PULTSFWA (Eq.(28)) and PULTSFWG (Eq.(29)) operators are used to aggregate the evaluation information to calculate the overall decision value of each alternative.

$$\tilde{\mathcal{F}}_i^A(p) = PULTSFWA(\tilde{\mathcal{g}}_{i1}(p), \tilde{\mathcal{g}}_{i2}(p), \ldots, \tilde{\mathcal{g}}_{in}(p))$$

$$= \left(\begin{array}{c} \left\{ \left[s_{k\sqrt[q]{1-\prod_{j=1}^{n}\left(1-\left(\mu_{ij(t)}^{L}/k\right)^{q}\right)^{\varpi_j}}}, s_{k\sqrt[q]{1-\prod_{j=1}^{n}\left(1-\left(\mu_{ij(t)}^{U}/k\right)^{q}\right)^{\varpi_j}}} \right] \middle| \sum_{j=1}^{n} \varpi_j \hat{p}_{ij(t)} \right\}, \\ \left\{ \left[s_{k\prod_{j=1}^{n}\left(\eta_{ij(r)}^{L}/k\right)^{\varpi_j}}, s_{k\prod_{j=1}^{n}\left(\eta_{ij(r)}^{U}/k\right)^{\varpi_j}} \right] \middle| \prod_{j=1}^{n} \left(\tilde{p}_{ij(r)}\right)^{\varpi_j} \right\}, \\ \left\{ \left[s_{k\prod_{j=1}^{n}\left(\nu_{ij(w)}^{L}/k\right)^{\varpi_j}}, s_{k\prod_{j=1}^{n}\left(\nu_{ij(w)}^{U}/k\right)^{\varpi_j}} \right] \middle| \prod_{j=1}^{n} \left(\bar{p}_{ij(w)}\right)^{\varpi_j} \right\} \end{array} \right) \quad (28)$$

and

$$\tilde{\mathcal{F}}_i^G(p) = PULTSFWG(\tilde{\mathcal{g}}_{i1}(p), \tilde{\mathcal{g}}_{i2}(p), \ldots, \tilde{\mathcal{g}}_{in}(p))$$

$$= \left(\begin{array}{c} \left\{ \left[s_{k\prod_{j=1}^{n}\left(\mu_{ij(t)}^{L}/k\right)^{\varpi_j}}, s_{k\prod_{j=1}^{n}\left(\mu_{ij(t)}^{U}/k\right)^{\varpi_j}} \right] \middle| \prod_{j=1}^{n} \left(\hat{p}_{ij(t)}\right)^{\varpi_j} \right\}, \\ \left\{ \left[s_{k\sqrt[q]{1-\prod_{j=1}^{n}\left(1-\left(\eta_{ij(r)}^{L}/k\right)^{q}\right)^{\varpi_j}}}, s_{k\sqrt[q]{1-\prod_{j=1}^{n}\left(1-\left(\eta_{ij(r)}^{U}/k\right)^{q}\right)^{\varpi_j}}} \right] \middle| \sum_{j=1}^{n} \varpi_j \tilde{p}_{ij(r)} \right\}, \\ \left\{ \left[s_{k\sqrt[q]{1-\prod_{j=1}^{n}\left(1-\left(\nu_{ij(w)}^{L}/k\right)^{q}\right)^{\varpi_j}}}, s_{k\sqrt[q]{1-\prod_{j=1}^{n}\left(1-\left(\nu_{ij(w)}^{U}/k\right)^{q}\right)^{\varpi_j}}} \right] \middle| \sum_{j=1}^{n} \varpi_j \bar{p}_{ij(w)} \right\} \end{array} \right) \quad (29)$$

Step 4. The values of $Sc(\tilde{\mathcal{F}}_i^A(p))$ and $Sc(\tilde{\mathcal{F}}_i^G(p))$ are evaluated from Eq. (7), respectively. Since the PULTSFWA and PULTSFWG operators possess distinct characteristics. The former being compensatory, emphasizing the independence of criteria and allowing for complementary of advantages, while the latter is non-compensatory, emphasizing synergy or interdependence among criteria and exhibiting a "short-board effect". The results obtained by these two aggregation operators may lead to the following two scenarios: 1) The ranking results of the alternatives are highly consistent, indicating that the superiority or inferiority among the alternatives is clear and the results are robust; 2) The ranking results of the alternatives show significant differences.

In response to scenario 2), we construct a comprehensive value function $C(\hbar_i)$ (Eq.(30)) to achieve a balance between the compensatory of the non-compensatory of the latter.

$$C(\hbar_i) = Sc(\tilde{\mathcal{F}}_i^A(p)) \oplus Sc(\tilde{\mathcal{F}}_i^G(p)) \tag{30}$$

Step 5. Arrange all options in descending order. The option with the highest score is the best choice.

6. Numerical example

In this section, we use a practical MCDM problem to illustrate the application of the developed method. Suppose an organization plans to implement an enterprise resource planning (ERP) system (adapted from Refs.[28,29]). The first step is to form a project team consisting of a chief information manager and two senior representatives from the user department. By collecting all possible information about ERP vendors and systems, the project team selected four potential ERP systems $H=\{\hbar_i|i=1,2,...,4\}$ as candidates. The company hired some external professional organizations or experts to assist in this decision-making process. The project team selected four attributes to evaluate the alternatives: (1) function and technology $\beth_1$, (2) strategic fit $\beth_2$, (3) supplier capability $\beth_3$, (4) supplier reputation $\beth_4$. In order to quantify each LTS S={ s_0= extremely poor, s_1= very poor, s_2= poor, s_3= medium, s_4=good, s_5= very good, s_6= extremely good}. DMs use PULTSFNs to evaluate four potential ERP systems under the above four criteria, and construct a matrix $\widetilde{\mathcal{D}}$ as shown in Table 1.

Table 1. Initial PULTSF decision matrix $\widetilde{\mathcal{D}}$

	$\beth_1$	$\beth_2$
$\hbar_1$	$\langle\{[s_2,s_4]\|0.2, [s_3,s_4]\|0.3, [s_4,s_5]\|0.5\},$ $\{[s_0,s_1]\|0.1, [s_1,s_3]\|0.8, [s_2,s_3]\|0.1\},$ $\{[s_0,s_1]\|0.5, [s_0,s_2]\|0.3, [s_1,s_3]\|0.2\}\rangle$	$\langle\{[s_4,s_5]\|0.4, [s_4,s_6]\|0.2, [s_5,s_6]\|0.4\},$ $\{[s_0,s_1]\|0.3, [s_1,s_2]\|0.5, [s_1,s_3]\|0.2\},$ $\{[s_0,s_2]\|0.6, [s_1,s_2]\|0.1, [s_2,s_3]\|0.3\}\rangle$
$\hbar_2$	$\langle\{[s_3,s_4]\|0.3, [s_4,s_5]\|0.6, [s_5,s_6]\|0.1\},$ $\{[s_1,s_3]\|0.7, [s_2,s_4]\|0.3\},$ $\{[s_0,s_1]\|0.5, [s_1,s_2]\|0.2, [s_2,s_3]\|0.2\}\rangle$	$\langle\{[s_1,s_3]\|0.6, [s_2,s_4]\|0.4\},$ $\{[s_3,s_4]\|0.2, [s_4,s_5]\|0.5, [s_5,s_6]\|0.3\},$ $\{[s_0,s_1]\|0.3, [s_1,s_2]\|0.2, [s_2,s_3]\|0.5\}\rangle$
$\hbar_3$	$\langle\{[s_3,s_4]\|0.1, [s_4,s_5]\|0.3, [s_5,s_6]\|0.6\},$ $\{[s_0,s_1]\|0.6, [s_1,s_2]\|0.3, [s_2,s_3]\|0.1\},$ $\{[s_0,s_2]\|0.9, [s_2,s_3]\|0.1\}\rangle$	$\langle\{[s_1,s_2]\|0.2, [s_2,s_3]\|0.5, [s_3,s_4]\|0.3\},$ $\{[s_2,s_3]\|0.3, [s_3,s_4]\|0.6, [s_4,s_5]\|0.1\},$ $\{[s_0,s_1]\|0.6, [s_1,s_3]\|0.4\}\rangle$
$\hbar_4$	$\langle\{[s_3,s_4]\|0.1, [s_4,s_5]\|0.6, [s_5,s_6]\|0.3\},$ $\{[s_0,s_1]\|0.2, [s_1,s_2]\|0.7, [s_2,s_3]\|0.1\},$ $\{[s_0,s_2]\|0.9, [s_2,s_3]\|0.1\}\rangle$	$\langle\{[s_2,s_3]\|0.1, [s_3,s_4]\|0.3, [s_4,s_5]\|0.6\},$ $\{[s_0,s_1]\|0.3, [s_1,s_3]\|0.7\},$ $\{[s_0,s_1]\|0.6, [s_1,s_2]\|0.3, [s_2,s_3]\|0.1\}\rangle$

	$\beth_3$	$\beth_4$
$\hbar_1$	$\langle\{[s_2,s_3]\|0.4, [s_2,s_4]\|0.4, [s_3,s_5]\|0.2\},$ $\{[s_1,s_3]\|0.6, [s_2,s_4]\|0.4\},$ $\{[s_0,s_1]\|0.4, [s_1,s_2]\|0.3, [s_2,s_3]\|0.3\}\rangle$	$\langle\{[s_0,s_2]\|0.2, [s_1,s_3]\|0.8\},$ $\{[s_3,s_5]\|0.4, [s_4,s_5]\|0.4, [s_5,s_6]\|0.2\},$ $\{[s_0,s_2]\|0.2, [s_1,s_2]\|0.4, [s_2,s_3]\|0.4\}\rangle$
$\hbar_2$	$\langle\{[s_0,s_1]\|0.6, [s_1,s_2]\|0.2, [s_2,s_3]\|0.2\},$ $\{[s_3,s_4]\|0.1, [s_4,s_5]\|0.7, [s_5,s_6]\|0.2\},$ $\{[s_0,s_2]\|0.8, [s_2,s_3]\|0.2\}\rangle$	$\langle\{[s_3,s_5]\|0.8, [s_5,s_6]\|0.2\},$ $\{[s_0,s_1]\|0.3, [s_1,s_2]\|0.5, [s_2,s_3]\|0.2\},$ $\{[s_0,s_1]\|0.8, [s_1,s_2]\|0.1, [s_2,s_3]\|0.1\}\rangle$
$\hbar_3$	$\langle\{[s_0,s_1]\|0.4, [s_1,s_2]\|0.4, [s_2,s_3]\|0.2\},$ $\{[s_3,s_5]\|0.5, [s_5,s_6]\|0.5\},$ $\{[s_0,s_1]\|0.1, [s_1,s_2]\|0.6, [s_2,s_3]\|0.3\}\rangle$	$\langle\{[s_2,s_3]\|0.2, [s_3,s_4]\|0.3, [s_4,s_5]\|0.5\},$ $\{[s_0,s_1]\|0.1, [s_1,s_2]\|0.6, [s_2,s_3]\|0.1\},$ $\{[s_0,s_1]\|0.7, [s_1,s_3]\|0.3\}\rangle$
$\hbar_4$	$\langle\{[s_0,s_1]\|0.5, [s_1,s_2]\|0.4, [s_2,s_3]\|0.1\},$ $\{[s_3,s_4]\|0.3, [s_4,s_5]\|0.5, [s_5,s_6]\|0.2\},$ $\{[s_0,s_1]\|0.3, [s_1,s_2]\|0.6, [s_2,s_3]\|0.1\}\rangle$	$\langle\{[s_0,s_1]\|0.8, [s_1,s_3]\|0.2\},$ $\{[s_2,s_3]\|0.1, [s_3,s_4]\|0.2, [s_4,s_5]\|0.7\},$ $\{[s_0,s_1]\|0.6, [s_1,s_2]\|0.3, [s_2,s_3]\|0.1\}\rangle$

6.1 Decision analysis

In order to select the most ideal ERP system, we use a method of over-criteria decision making based on PULTSF information. The specific steps are as follows :

Step 1. According to Definition 9, the PULTSFNs in Table 1 are normalized. Since there is no cost type in these criteria, there is no need for Eq. (24) to be over-transformed, and the standardized PULTSF decision matrix $\tilde{G}$ can be obtained, see Table 2. Through calculation, the value of parameter q is not less than 3, and the value of this paper is q = 3.

Table 2. Standardized PULTSF decision matrix $\tilde{G}$

<table>
<tr><th></th><th>$\beth_1$</th><th>$\beth_2$</th></tr>
<tr><td>$\hbar_1$</td><td><{[s_2,s_4]|0.2, [s_3,s_4]|0.3, [s_4,s_5]|0.5},
{[s_0,s_1]|0.1, [s_1,s_2]|0.4, [s_2,s_3]|0.5},
{[s_0,s_1]|0.65, [s_1,s_2]|0.25, [s_2,s_3]|0.1}></td><td><{[s_4,s_5]|0.4, [s_4,s_6]|0.2, [s_5,s_6]|0.4},
{[s_0,s_1]|0.3, [s_1,s_2]|0.6, [s_2,s_3]|0.1},
{[s_0,s_1]|0.3, [s_1,s_2]|0.4, [s_2,s_3]|0.3}></td></tr>
<tr><td>$\hbar_2$</td><td><{[s_3,s_4]|0.3, [s_4,s_5]|0.6, [s_5,s_6]|0.1},
{[s_1,s_2]|0.35, [s_2,s_3]|0.5 ,[s_3,s_4]|0.15},
{[s_0,s_1]|0.556,[s_1,s_2]|0.222,[s_2,s_3]|0.222></td><td><{[s_1,s_2]|0.3, [s_2,s_3]|0.5, [s_3,s_4]|0.2},
{[s_3,s_4]|0.2, [s_4,s_5]|0.5, [s_5,s_6]|0.3},
{[s_0,s_1]|0.3, [s_1,s_2]|0.2, [s_2,s_3]|0.5}></td></tr>
<tr><td>$\hbar_3$</td><td><{[s_3,s_4]|0.1, [s_4,s_5]|0.3, [s_5,s_6]|0.6},
{[s_0,s_1]|0.6, [s_1,s_2]|0.3, [s_2,s_3]|0.1},
{[s_0,s_1]|0.45, [s_1,s_2]|0.45, [s_2,s_3]|0.1}></td><td><{[s_1,s_2]|0.2, [s_2,s_3]|0.5, [s_3,s_4]|0.3},
{[s_2,s_3]|0.3, [s_3,s_4]|0.6, [s_4,s_5]|0.1},
{[s_0,s_1]|0.6, [s_1,s_2]|0.2, [s_2,s_3]|0.2}></td></tr>
<tr><td>$\hbar_4$</td><td><{[s_3,s_4]|0.1, [s_4,s_5]|0.6, [s_5,s_6]|0.3},
{[s_0,s_1]|0.2, [s_1,s_2]|0.7, [s_2,s_3]|0.1},
{[s_0,s_1]|0.45, [s_1,s_2]|0.45, [s_2,s_3]|0.1}></td><td><{[s_2,s_3]|0.1, [s_3,s_4]|0.3, [s_4,s_5]|0.6},
{[s_0,s_1]|0.3,[s_1,s_2]|0.35,[s_2,s_3]|0.35},
{[s_0,s_1]|0.6, [s_1,s_2]|0.3, [s_2,s_3]|0.1}></td></tr>
<tr><th></th><th>$\beth_3$</th><th>$\beth_4$</th></tr>
<tr><td>$\hbar_1$</td><td><{[s_2,s_3]|0.4, [s_2,s_4]|0.4, [s_3,s_5]|0.2},
{[s_1,s_2]|0.3, [s_2,s_3]|0.5, [s_3,s_4]|0.2},
{[s_0,s_1]|0.4, [s_1,s_2]|0.3, [s_2,s_3]|0.3}></td><td><{[s_0,s_1]|0.1, [s_1,s_2]|0.5, [s_2,s_3]|0.4},
{[s_3,s_4]|0.2, [s_4,s_5]|0.6, [s_5,s_6]|0.2},
{[s_0,s_1]|0.1, [s_1,s_2]|0.5, [s_2,s_3]|0.4}></td></tr>
<tr><td>$\hbar_2$</td><td><{[s_0,s_1]|0.6, [s_1,s_2]|0.2, [s_2,s_3]|0.2},
{[s_3,s_4]|0.1, [s_4,s_5]|0.7, [s_5,s_6]|0.2},
{[s_0,s_1]|0.4, [s_1,s_2]|0.4, [s_2,s_3]|0.2}></td><td><{[s_3,s_4]|0.4, [s_4,s_5]|0.4, [s_5,s_6]|0.2},
{[s_0,s_1]|0.3, [s_1,s_2]|0.5, [s_2,s_3]|0.2},
{[s_0,s_1]|0.8, [s_1,s_2]|0.1, [s_2,s_3]|0.1}></td></tr>
<tr><td>$\hbar_3$</td><td><{[s_0,s_1]|0.4, [s_1,s_2]|0.4, [s_2,s_3]|0.2},
{[s_3,s_4]|0.25, [s_4,s_5]|0.25, [s_5,s_6]|0.5},
{[s_0,s_1]|0.1, [s_1,s_2]|0.6, [s_2,s_3]|0.3}></td><td><{[s_2,s_3]|0.2, [s_3,s_4]|0.3, [s_4,s_5]|0.5},
{[s_0,s_1]|0.125,[s_1,s_2]|0.75,[s_2,s_3]|0.125},
{[s_0,s_1]|0.7, [s_1,s_2]|0.15, [s_2,s_3]|0.15}></td></tr>
<tr><td>$\hbar_4$</td><td><{[s_0,s_1]|0.5, [s_1,s_2]|0.4, [s_2,s_3]|0.1},
{[s_3,s_4]|0.3, [s_4,s_5]|0.5, [s_5,s_6]|0.2},
{[s_0,s_1]|0.3, [s_1,s_2]|0.6, [s_2,s_3]|0.1}></td><td><{[s_0,s_1]|0.8,[s_1,s_2]|0.1 , [s_2,s_3]|0.2},
{[s_2,s_3]|0.1, [s_3,s_4]|0.2, [s_4,s_5]|0.7},
{[s_0,s_1]|0.6, [s_1,s_2]|0.3, [s_2,s_3]|0.1}></td></tr>
</table>

Step 2. The weight vector of each criteria is calculated by using the Eqs. (25~27). $\varpi = (0.275, 0.214, 0.234, 0.276)^{\mathrm{T}}$.

Step 3. The Eqs.(28~29) are used to calculate the overall decision value $\tilde{\mathcal{F}}_i(p)$ of each alternative, respectively, as shown in Table 3.

Step 4. The overall value of each alternative is refined by using Eq.(7), and the score function value is presented in Table 3. The ranking result of alternatives obtained by PULTSFWA operator is : $\hbar_1 \succ \hbar_3 \succ \hbar_2 \succ \hbar_4$; the ranking result of the alternatives obtained by the PULTSFWG operator is : $\hbar_3 \succ \hbar_1 \succ \hbar_4 \succ \hbar_2$. Evidently, the ranking results of the alternatives obtained by the two AOs are inconsistent. The comprehensive value $C(\hbar_i)$ for each alternative is subsequently calculated using Eq.(30).

$$C(\hbar_1) = Sc(\tilde{\mathcal{F}}_1^A(p)) \oplus Sc(\tilde{\mathcal{F}}_1^G(p)) = s_{0.5616} \oplus s_{0.4782} = s_{0.5616+0.4782-\frac{0.5616\times 0.4782}{7}} = s_{1.0014}$$

Similarly, we derive: $C(\hbar_2) = s_{0.9567}$, $C(\hbar_3) = s_{1.0078}$, and $C(\hbar_4) = s_{0.9498}$.

Step 5. Rank the options in descending order, $\hbar_3 \succ \hbar_1 \succ \hbar_2 \succ \hbar_4$. Thus, $\hbar_3$ is the best option.

Table 3. The overall decision value and the score function value of each alternative

PULTSFWA	$\tilde{\mathcal{F}}_i(p)$	$Sc(\tilde{\mathcal{F}}_i^A(p))$

$\hbar_1$	$<\{[s_{2.658},s_{3.550}]\vert 0.262, [s_{2.893},s_{4.565}]\vert 0.357, [s_{3.835},s_{5.038}]\vert 0.381\},$ $\{[s_{0.000},s_{1.725}]\vert 0.198, [s_{1.725},s_{2.833}]\vert 0.514, [s_{2.833},s_{3.887}]\vert 0.222\},$ $\{[s_{0.000},s_{1.000}]\vert 0.293, [s_{1.000},s_{2.000}]\vert 0.350, [s_{2.000},s_{3.000}]\vert 0.240\}>$	$s_{0.5616}$
$\hbar_2$	$<\{[s_{2.48},s_{3.382}]\vert 0.398, [s_{3.382},s_{4.344}]\vert 0.430, [s_{4.433},s_{5.393}]\vert 0.172\},$ $\{[s_{0.000},s_{2.254}]\vert 0.222, [s_{2.254},s_{3.373}]\vert 0.541, [s_{3.373},s_{4.431}]\vert 0.202\},$ $\{[s_{0.000},s_{1.000}]\vert 0.499, [s_{1.000},s_{2.000}]\vert 0.200, [s_{2.000},s_{3.000}]\vert 0.207\}>$	$s_{0.5423}$
$\hbar_3$	$<\{[s_{2.157},s_{3.037}]\vert 0.219, [s_{3.037},s_{3.90}]\vert 0.366, [s_{3.990},s_{5.025}]\vert 0.414\},$ $\{[s_{0.000},s_{1.751}]\vert 0.273, [s_{1.751},s_{2.876}]\vert 0.430, [s_{2.876},s_{3.937}]\vert 0.155\},$ $\{[s_{0.000},s_{1.000}]\vert 0.380, [s_{1.000},s_{2.000}]\vert 0.299, [s_{2.000},s_{3.000}]\vert 0.168\}>$	$s_{0.5599}$
$\hbar_4$	$<\{[s_{2.106},s_{2.926}]\vert 0.387, [s_{2.926},s_{3.844}]\vert 0.351, [s_{3.844},s_{4.876}]\vert 0.290\},$ $\{[s_{0.000},s_{1.874}]\vert 0.198, [s_{1.874},s_{3.002}]\vert 0.394, [s_{3.002},s_{4.064}]\vert 0.263\},$ $\{[s_{0.000},s_{1.000}]\vert 0.471, [s_{1.000},s_{2.000}]\vert 0.394, [s_{2.000},s_{3.000}]\vert 0.100\}>$	$s_{0.5311}$
PULTSFWG	$\tilde{\mathcal{F}}_i(p)$	$Sc\left(\tilde{\mathcal{F}}_i^G(p)\right)$
$\hbar_1$	$<\{[s_{0.000},s_{2.471}]\vert 0.225, [s_{2.471},s_{3.603}]\vert 0.339, [s_{3.239},s_{4.515}]\vert 0.362\},$ $\{[s_{1.993},s_{2.771}]\vert 0.217, [s_{2.771},s_{3.673}]\vert 0.522, [s_{3.673},s_{4.709}]\vert 0.261\},$ $\{[s_{0.000},s_{1.000}]\vert 0.364, [s_{1.000},s_{2.000}]\vert 0.363, [s_{2.000},s_{3.000}]\vert 0.273\}>$	$s_{0.4782}$
$\hbar_2$	$<\{[s_{0.000},s_{2.492}]\vert 0.382, [s_{2.492},s_{3.616}]\vert 0.399, [s_{3.616},s_{4.676}]\vert 0.165\},$ $\{[s_{2.330},s_{3.195}]\vert 0.245, [s_{3.5},s_{4.143}]\vert 0.547, [s_{4.143},s_{5.194}]\vert 0.208\},$ $\{[s_{0.000},s_{1.000}]\vert 0.532, [s_{1.000},s_{2.000}]\vert 0.225, [s_{2.000},s_{3.000}]\vert 0.243\}>$	$s_{0.4495}$
$\hbar_3$	$<\{[s_{0.000},s_{2.302}]\vert 0.194, [s_{2.302},s_{3.400}]\vert 0.358, [s_{3.400},s_{4.447}]\vert 0.380\},$ $\{[s_{2.018},s_{2.819}]\vert 0.322, [s_{2.819},s_{3.726}]\vert 0.477, [s_{3.726},s_{4.751}]\vert 0.201\},$ $\{[s_{0.000},s_{1.000}]\vert 0.469, [s_{1.000},s_{2.000}]\vert 0.349, [s_{2.000},s_{3.000}]\vert 0.182\}>$	$s_{0.4868}$
$\hbar_4$	$<\{[s_{0.000},s_{1.853}]\vert 0.259, [s_{1.853},s_{2.986}]\vert 0.287, [s_{2.986},s_{4.050}]\vert 0.241\},$ $\{[s_{2.057},s_{2.883}]\vert 0.217, [s_{2.883},s_{3.803}]\vert 0.440, [s_{3.803},s_{4.829}]\vert 0.343\},$ $\{[s_{0.000},s_{1.000}]\vert 0.489, [s_{1.000},s_{2.000}]\vert 0.411, [s_{2.000},s_{3.000}]\vert 0.100\}>$	$s_{0.4531}$

6.2 Sensitivity analysis

we need to test the influence of different values of parameter q on the criteria weight value and the final alternative ranking. The parameter q takes nine values in the interval of 3 to 19, and we find that as the value of the q becomes larger, the weight values of each criteria also have different trends. As shown in Figure 1, the weight ϖ_1 of criteria $\beth_1$ increases with the increase of q value, while the weight ϖ_2 of criteria $\beth_2$ has the opposite trend, and the weights of criteria $\beth_3$ and $\beth_4$ are relatively stable. This shows that the change of parameter q can adjust the configuration of all criteria weight values, and the higher parameter q value can lead to the change of criteria weight vector.

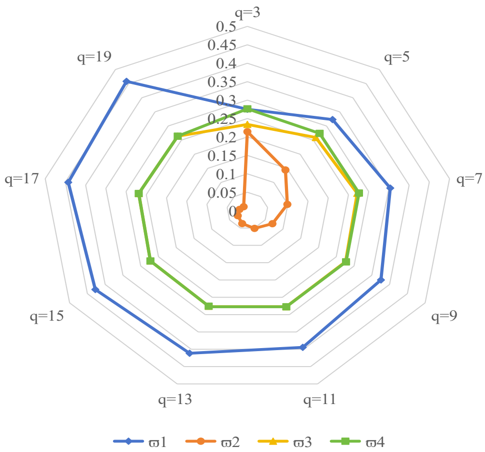

Fig 1. Variation of criterion weights with q

For the influence of the parameter q value on the ranking of each alternative, we also use thirteen values of the parameter q in the interval of 3 to 15. We obtain the final values of each alternative with different parameter q values under two AOs. The trends of these values are shown in Figure 2 and Figure 3, respectively. It can be found that in Figure 2, when the PULTSFWA operator is used to aggregate PULTSFNs under each criterion, the overall final value of each alternative is also decreasing with the increase of parameter q, and approaches $s_{0.500}$. At this time, the ranking changes of each alternative are shown in Figure 4. When q = 3, the alternative $\hbar_1$ is in the first place, but with the increase of q value, its ranking gradually lags behind, while the alternative $\hbar_3$ is stable in the first place when q≥4.

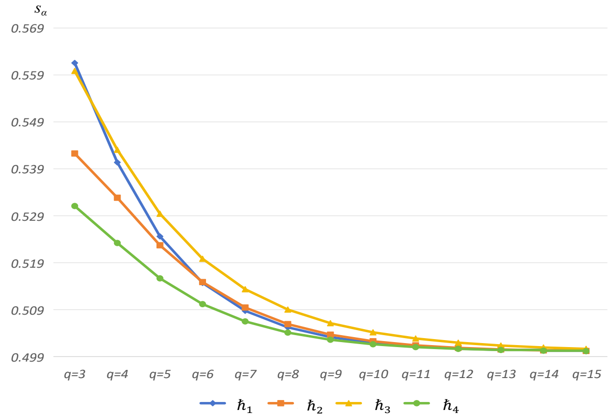

Fig 2. Decrease of alternative scores versus increasing *q*

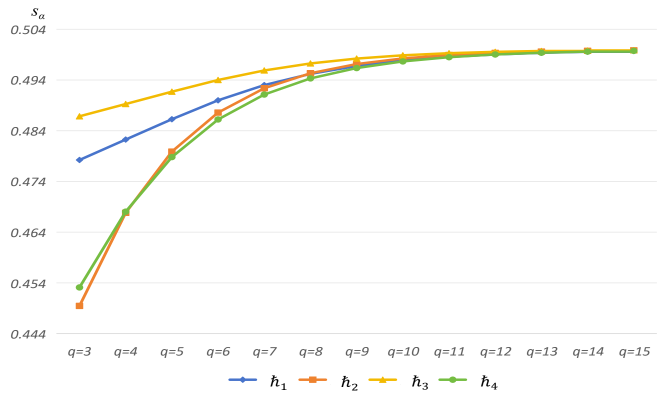

Fig 3. The score values of alternative increase with *q*

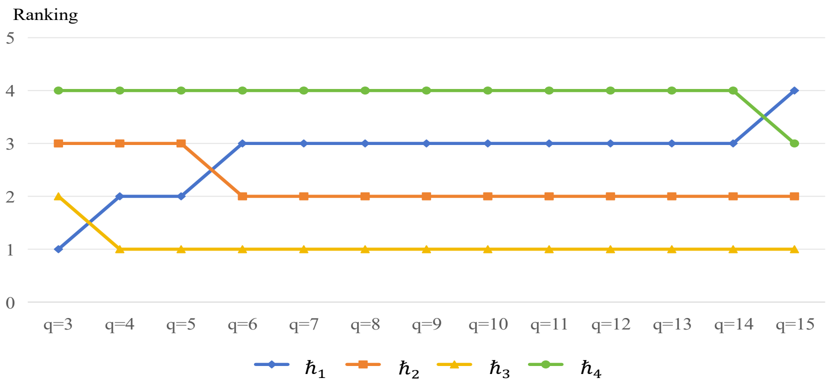

Fig 4. Alternative rankings versus *q*

In Figure 3, when using the PULTSFWG operator to aggregate PULTSFNs under each criteria, as the parameter q increases, the overall final value of each scheme is also increasing and approaching $s_{0.500}$. At this time, the ranking changes of each alternative are shown in Figure 5. In the process of q value change, the alternative $\hbar_3$ always ranks first, while the alternative $\hbar_4$ ranks from the last row to the second row, and the alternative $\hbar_1$ ranks from the second row to the last row. In general, regardless of the change of q value, the optimal alternative is stable at $\hbar_3$. We can determine the q

value according to the DMs' preference to select the best alternative, which shows the reliability of the method proposed in this paper.

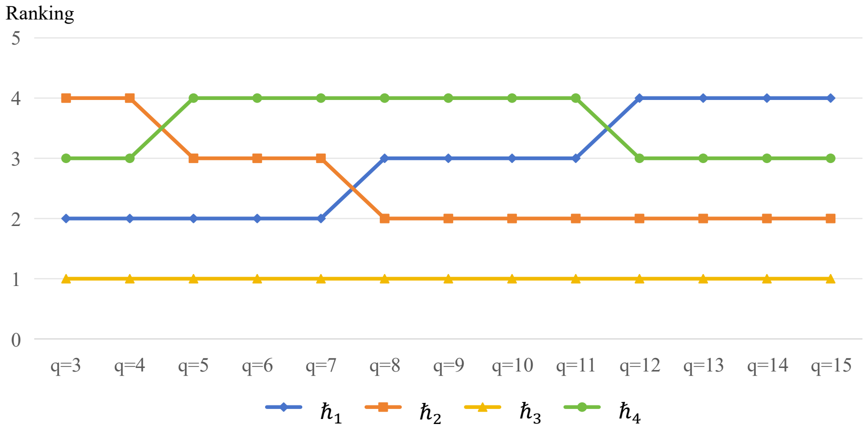

Fig 5. Alternative rankings versus *q*

6.3 Comparison analysis

In this section, we introduce some existing decision methods to compare the decision problems in the examples with the methods proposed in this paper, such as PL*q*-ROFWA[26], PUL*q*-ROFWA[19], PUL*q*-ROFWG[19], PLTSFWPA[25], PLTSFWPG[25], IVPLTSFWA[30] and IVPLTSFWG[30]. However, these methods are not applicable to the examples in this paper, because PL*q*-ROFS, PUL*q*-ROFS, PLTSFS and IVPLTSFS cannot process the evaluation information in the PULTSFS environment. Table 4 shows the comparison of the characteristics of different AOs.

Table 4. Comparison of characteristics of different AOs

AOs	Dimension	Linguistic element	Generality	Computational intensity	Probability calculation
PL*q*-ROFWA[26]	2-dimensions	LTS	Low	Moderate	Pre-adjustment
PUL*q*-ROFWA[19]	2-dimensions	ULV	Medium	Simple	Pre-adjustment
PUL*q*-ROFWG[19]	2-dimensions	ULV	Medium	Simple	Pre-adjustment
PLTSFWPA[25]	3-dimension	LTS	Low	Moderate	Direct multiplication, without considering the weight
PLTSFWPG[25]	3-dimension	LTS	Low	Moderate	Direct multiplication, without considering the weight
IVPLTSFWA[30]	3-dimension	LTS	Medium	Complex	Direct multiplication, without considering the weight
IVPLTSFWG[30]	3-dimension	LTS	Medium	Complex	Direct multiplication, without considering the weight
PULTSFWA	3-dimension	ULV	High	Simple	Consider weight
PULTSFWG	3-dimension	ULV	High	Simple	Consider weight

The Table 4 presents these AOs derived from distinct decision-making contexts: PL*q*-ROFS, PUL*q*-ROFS, PLTSFS, IVPLTSFS and PULTSFS. They characterize evaluation information across different dimensions. For instance, PL*q*-ROFS and PUL*q*-ROFS express DMs' evaluation preferences through membership and non-membership degrees; whereas PLTSFS, IVPLTSFS, and PULTSFS construct evaluation information across 3-dimensions: membership, neutrality, and non-membership. Within each dimension, linguistic elements constitute crucial components, categorised here as either LTS or ULV. Compared to LTS representations, ULV approaches convey richer uncertainty information, aligning more closely with human cognition and judgement regarding phenomena. In terms of the generality of information representation, PLq-ROFS, PULq-ROFS, and PLTSFS constitute degenerate special forms of PULTSFS. Consequently, the PULTSFWA and PULTSFWG operators proposed herein exhibit a high degree of generality. In the computation of semantic elements, the PLq-ROFWA, PLTSFWPA, and PLTSFWPG operators employ inverse functions for transformation, a process that introduces increased computational complexity. Furthermore, the IVPLTSFWA and IVPLTSFWG operators utilise union operations; however, as the number of elements within the set increases, so does the computational workload and complexity. Regarding probability calculation, the PLq-ROFWA, PULq-ROFWA, and PULq-ROFWG operators pre-assign the aggregated values to identical probability distributions, resulting in negligible probabilistic participation during aggregation. While this approach simplifies computation, the diversity of pre-set probability distributions directly influences aggregation outcomes. In the PLTSFWPA, PLTSFWPG, IVPLTSFWA, and IVPLTSFWG operators, probability calculations employ a multiplicative form regardless of whether arithmetic or geometric averaging is applied. Consequently, the respective importance of experts or criteria is not emphasised. Moreover, as the number of inputs increases during aggregation, the resulting probability approaches zero. In summary, the advantages of the AOs proposed in this paper are primarily manifested in the following three aspects:

The PULTSFS defined herein comprehensively characterize the hesitancy, uncertainty, probabilistic nature, and fuzziness of evaluation information across three dimensions: membership degree, neutrality, and non-membership degree. Its generality manifests in the fact that PULTSFS can degenerate into distinct specialised forms under varying contexts, such as PLq-ROFS[26], PULq-ROFS[19] and PLTSFS[25] thereby accommodating a broader spectrum of practical decision-making scenarios. Consequently, the PULTSFWA and PULTSFWG operators developed upon the PULTSFS framework exhibit greater generality and sophistication compared to existing alternative methodologies.

The PULTSFWA and PULTSFWG operators developed herein constitute computational integrations of operations divided into uncertain linguistic and probabilistic components. Compared to the PULTS operation rules proposed by Lin et al.[13], the proposed operator rules achieve closure for the AOs. Furthermore, the aggregation operators presented herein avoid the partial information loss and computational complexity associated with the inverse function transformation operation [17, 31].

During the AO operation, we considered the relative importance of both uncertain linguistic and probabilistic information. By integrating weight values during the aggregation process, this avoids the probability becoming excessively small during the probability multiplication process [25] thereby ensuring it accurately reflects the semantic probability distribution following the fusion of evaluation information. Simultaneously, this eliminates issues such as the subjectivity, arbitrariness, and diversity of results arising from prior probability adjustments[19, 26].

7. Conclusions

Owing to the subjective and ambiguous nature of human evaluations of objects, existing theoretical approaches such as fuzzy sets and LTSs cannot comprehensively and accurately represent and address genuinely complex decision-making problems. To this end, this paper proposes a powerful new concept termed the PULTSFS, building upon PULT and TSFS, to advance existing PLTS. The PULTSFS is defined as a three-dimensional mathematical structure comprising membership, neutrality, and non-membership dimensions, with information evaluated using PULTS on each dimension. To facilitate PULTSFS computation, its normalization process is investigated, establishing fundamental concepts including its scoring function, accuracy function, comparison rules, and the Hamming distance measure for PULTSFS. Subsequently, the paper establishes fundamental operational rules for PULTSFSs, upon which the PULTSFWA, PULTSFOWA, PULTSFWG, and PULTSFOWG operators are developed. The monotonicity, idempotency, and boundedness properties of these aggregation operators are analyzed. Subsequently, a PULTSF decision model based on the AOs is constructed to address MCDM problems where criteria weight information is entirely unknown. Finally, the validity and applicability of the method proposed in this paper are demonstrated through an example.

The PULTSFS constitutes a comprehensive, precise, and robust tool capable not only of capturing the uncertainty and ambiguity inherent in evaluative information within complex decision-making environments, but also of characterizing the hesitancy and potentiality associated with such uncertain linguistic expressions. As all three dimensions are measured using PULTSs, this linguistic modelling tool preserves more detailed decision information compared to existing PLTSs. Consequently, it yields more rational, comprehensive, and objective decision outcomes. The PULTSFS represents a novel advancement in PLTS, offering fresh methodologies for modelling uncertain language in MCDM decision-making and enriching the substance of decision theory and methodology. However, compared to existing PLTSs and their derivative approaches, the PULTSFS exhibits greater mathematical complexity and computational demands.

Future research will build upon this work to further investigate information measures related to PULTSFS, such as entropy, cross-entropy, and similarity. We will also explore novel operational rules that account for interactions among the three dimensions and correlations between PULTSFS variables. Furthermore, we aim to extend traditional decision-making techniques within this framework, including methods like TOPSIS, VIKOR and TODIM, applying these to practical decision-making problems in domains such as supply chain management, investment decisions and pattern recognition.

Author Contributions

Conceptualization, H.W.; methodology, H.W. and J.H.; formal analysis, H.W.; investigation, H.W. and H.J.; resources, H.W.; writing--original draft preparation, H.W. and H.J.; writing--review and editing, H.J. and K.Z.; visualization, K.Z.; supervision, H.W.; project administration, H.W.; funding acquisition, H.W. All authors have read and agreed to the published version of the manuscript.

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Data Availability Statement

Not applicable.

Conflicts of Interest

The authors declare no conflict of interest.

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